Liquidity Constraints in a Monetary Economy*

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Abstract

This paper presents a microfounded model of money with a consumption and an investment market. We consider an economy in which only part of the investment returns can be pledged. A liquidity constraint arises when the pledgeable part of the returns are not enough to pay for investment costs. We show that when the liquidity constraint is binding, agents may make a cash downpayment and money can perform two roles – as a provider of liquidity services and exchange services. The liquidity constraint constitutes a channel though which under-investment occurs even at low inflation rates.

Keywords: Liquidity, Money, Search

JEL: E40

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1 Introduction

Money is the medium used to transfer resources on the spot, while liquidity refers to the availability of a medium to transfer resources over time. The monetary search literature initiated by Kiyotaki and Wright (1989) has been successful in providing a solid micro-foundation based on trade frictions for the emergence of money as a medium of exchange. On the other hand, a recent growing literature emphasizes the importance of financial frictions and liquidity constraints for the emergence of a medium to transfer resources over time. In particular, Kiyotaki and Moore (2001b) study the effect of limited supply of liquid assets on investment. Although, intuitively, money and liquidity would seem to be linked, these two approaches take them as separate issues. The objective of the present paper is to explore a simple framework using a standard monetary search approach that allows us to study the issue of liquidity and its effect on investment. We are particularly interested in the relationship between money as a medium of spot trade and a medium of trade over time. Following Kiyotaki and Wright (1989), we assume that there exist frictions in spot trade. We introduce the notion of pledgeability and consider the possibility that the fundamental impediment arising in spot trade seeps into the credit market and hinders trade over time. In such an economy, agents may use money as a means of financing investment and money can perform two roles, as a provider of exchange and liquidity services.

Specifically, we consider a version of the divisible money model developed by Lagos and Wright (2005) which has a consumption and an investment market. Trading on the consumption market is subject to randomness and is not observable, hence money is used to lubricate the exchange of consumption goods. Trading on the investment market is instead frictionless. However, part of the investment returns accrue randomly to agents while they are trading on the opaque consumption market, and these returns cannot be pledged to outside investors to pay for investment costs. Thus, liquidity constraints may ensue. Within this setup, we show that when the average productivity of the returns is large enough to cover investment costs, the investment project is self-financing and money is used on the consumption but not on the investment market. Money is used as a medium of exchange but not as a provider of liquidity. In this case, equilibrium displays a dichotomous nature: agents make an investment decision independently of liquidity concerns and the equilibrium investment is at the optimal level from purely productive point of view. Thus, inflation generates distortions only in terms of consumption. On the contrary, when the average productivity of the returns is relatively small, liquidity constraints arise in equilibrium and agents under-invest. In this case, agents use money both to relax the liquidity constraint and to finance consumption, thus inflation generates distortions both in terms of investment and consumption – a relationship which turns out to be complementary. However, for sufficiently high inflation rates, money becomes relatively useless as a provider of liquidity services and agents stop using it to finance investment.

Our paper shares features with Kiyotaki and Wright (1989) and Kiyotaki and Moore (2001a), placing – so to speak – the former at the heart of the latter. The notion of the inability to pledge the entire returns of a project, which is related to the one we have adopted in the present paper, occupies the central stage in a series of papers by Nobuhiro Kiyotaki and John Hardman Moore. An entrepreneur, they argue, can issue claims to an investor only up to a certain fraction of his future returns, for instance, because of moral hazard reasons. Could the investor spend such claims with a third party, extraneous to the initial deal, instead of having to hold on to them till the project pays off, he would be keener to lend in the first place. A circulating, money-like instrument which is held for its transaction value rather than its maturity value, may emerge in a world where individuals cannot trust each other to keep their promises. In contrast, we have taken the point of view that money already exists in society as a medium of spot trade. In such a world, we argue, the frictions hindering trade – such as randomness and opacity of transactions – may affect the ability of agents to pledge returns to outsiders.

The definition of liquidity we adopt in this paper is akin to the one used in the non-monetary model of Holmstrom and Tirole (1998), where moral hazard is responsible for the limited pledgeability of returns. The non-monetary model by Gertler and Rogoff (1990) also features investment market imperfections which arise endogenously. In our model, liquidity issues are linked to the role of money as a medium of exchange, and arise from frictions of the search type.

The closest paper in the monetary search literature is perhaps Telyukova and Wright (2008), who introduce credit in a Lagos and Wright (2005) framework. Repayment is perfectly enforceable in their model, while in the present context it can be guaranteed only up to a point. Other related papers *à* la Lagos and Wright (2005) are Berentsen, Camera and Waller (2008), where some agents can lend their otherwise idle money holdings, Lagos and Rocheteau (2008) featuring capital as an alternative medium of exchange, Ferraris and Watanabe (2008) where capital can serve as collateral to secure the repayment of debt, Dong (2009) where money and costly credit are analyzed, and Aruoba, Waller and Wright (2008) where capital can be used to produce the consumption good. The papers by Duffie, Garleanu and Pedersen (2005) and Lagos and Rocheteau (2009) study liquidity issues in a search model but from a different perspective, in a framework where trade is mediated by specialists.

Finally, the paper relates to the literature featuring a cash-in-advance (CIA) model with investment, where the CIA constraint applies to both consumption and investment purchases, as in Stockman (1981), generating a negative impact of inflation on investment. In our paper, agents choose whether to pay with cash, with part of their future returns or a mix of the two. When agents opt for cash, under-investment ensues, but the possibility of partly compensating with future returns mitigates the distortion, relative to Stockman (1981).

The rest of the paper unfolds as follows. Section 2 presents the baseline model and provides discussion and extension. Section 3 concludes. All the proofs are in the Appendix.

2 The Model

2.1 The Environment

We use a competitive version of the divisible money model developed by Lagos and Wright (2005). Time is discrete and continues for ever. At the start of each period the economy is inhabited by a $[0, 1]$ continuum of homogeneous entrepreneurs and a $[0, 1]$ continuum of homogeneous investors. As will be detailed shortly, the entrepreneurs are the active group of agents and the investors are the passive group in our economy. Each period is divided into three sub-periods: morning, afternoon and evening. Agents discount future payoffs at a rate $\beta \in (0, 1)$ across periods, but there is no discounting between the three sub-periods. A market is open in each sub-period. The marginal costs of all the production are measured in terms of utility, and we normalize all the marginal costs to be one. Economic activities in each sub-period are illustrated in Figure 1 and unfold as follows.

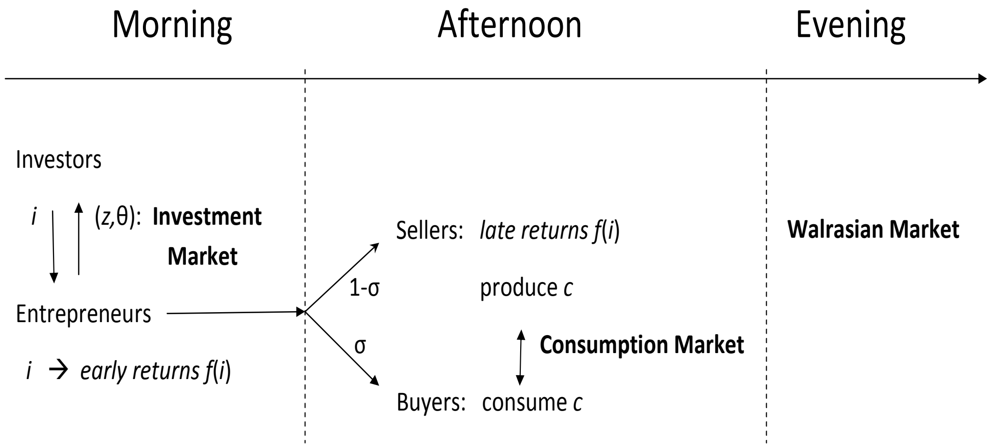

Figure 1: Timing of events

At the beginning of each morning, each investor produces *an investment good.* During the morning each entrepreneur is randomly matched to one investor. An entrepreneur offers a contract to an investor to buy the investment good. We will be more specific about the terms of contracts below. The investment good is worth zero in the hands of the investor, but once in the hands of an entrepreneur it can generate a perishable output. An investment good i yields output $f(i)$ at the end of both morning and afternoon within a given period. The function $f(\cdot)$ is twice continuously differentiable and strictly increasing and concave in its argument. It satisfies $f(0) = 0$, $f'(0) = \infty$ and $f'(\infty) = 0$. In what follows we shall refer to the morning output as *early return* and the afternoon output as *late return* of the investment. The early return is deterministic while the late return is stochastic as described below. Investment is a one-period event and the investment good fully decays at the end of the afternoon. The entrepreneurs derive linear utility from consuming these returns.

After the day market has closed, another market opens during the afternoon. In this market entrepreneurs can exchange among each other a perishable good, referred to as a *consumption good.* There exists also an intrinsically worthless good, which is perfectly divisible and storable, called fiat money. Trade in the afternoon market is subject to frictions. In the spirit of the monetary search model of Kiyotaki and Wright (1989), we model such frictions with two main ingredients. First, trade in the afternoon market is anonymous, thus the trading histories of agents are private knowledge. This implies, among other things, that investors cannot observe the activities of individual entrepreneurs during the afternoon. Second, entrepreneurs face randomness in their preferences and production possibilities. At the beginning of each afternoon, an entrepreneur is selected to be either a buyer or a seller. The former event happens with probability $\sigma \in (0,1)$ and the latter happens with probability $1-\sigma$. A seller does not wish to consume the consumption goods but is able to produce and sell them on the market. At the same time, a seller's production ability implies that he has access to the technology $f(\cdot)$ as well, hence a seller has an opportunity to consume the late return of investment. A buyer does not have access to the production technology but wishes to consume the consumption goods.[1] We denote by $u(c)$ the utility function for consumption of goods c. The function $u(\cdot)$ is twice continuously differentiable, strictly increasing and concave in its argument, and satisfies $u'(0) = \infty$ and $u'(\infty) = 0$. With no ability to access the technology $f(\cdot)$, the buyer does not have the opportunity to consume the late return of the investment and the investment good decays in the afternoon. The consumption market is competitive and agents take the market price, denoted by p, as given.

During the evening there is another opportunity for production. Agents can produce output with non-contractible effort. The evening market is walrasian and the output is traded at a per unit price normalized to unity. Fiat money can be traded for the output on this market at a price, denoted by ϕ, per unit.

The assumptions described above, i.e., the random buyer/seller division and the anonymity of transactions, are sufficient to ensure an essential role of money as a medium of exchange in the afternoon market: the sellers must receive money for immediate compensation of their products (i.e., consumption goods). The supply of fiat money is controlled by the government so that $M = \pi M_{-1}$, where M denotes the money stock at a given period and π denotes the gross growth rate of the money supply which we assume to be constant. Subscript -1 (or $+1$) stands for the previous (or next) period. New money is injected, or withdrawn, at the end of each period in the form of lump-sum transfers or taxes by an amount denoted by τ. All agents receive transfers or are taxed equally.

[1] This specification is for simplicity and does not affect the main results – one can assume that the late return is deterministic, rather than stochastic. Later in the paper, we explore a different specification of the model, where a seller uses the investment good to produce the consumption good during the afternoon.

2.2 Equilibrium

Before describing equilibria, we shall first remark the efficient allocation, denoted by i^*, c^*, which satisfies the following conditions

$$f'(i^*) + (1-\sigma)f'(i^*) = 1 \tag{1}$$

$$u'(c^*) = 1. \tag{2}$$

(1) equates the total expected marginal returns, measured in terms of utility, of the investment goods to its total marginal costs (= 1). (2) equates the total marginal utility of the consumption goods to its total marginal costs.

In what follows, we construct symmetric steady state monetary equilibria where agents of identical type take identical strategies, all real variables are constant over time and money is valued (i.e. $\phi > 0$). We consider monetary policies such that $\pi \geq \beta$ and when $\pi = \beta$ (which is the Friedman rule) we only consider the limiting equilibrium as the rate of inflation π approaches to the discount factor β. In this section the emphasis is placed on the intuition behind the equilibrium conditions. The formal derivation can be found in the Appendix.

At the start of each period, each entrepreneur offers to a randomly assigned investor a contract which involves a payment out of future resources in exchange for an amount of investment goods. The environment described above has two important implications for the contracts. First, long term contracts are not available because of the random matching process in a large economy: there is no chance for a matched pair to meet each other again at any future periods. Second, the presence of informational frictions in the afternoon market implies that the late return of investment cannot be pledged to outside investors. This is because the outcome of the afternoon market accrues privately to individual entrepreneurs and investors cannot observe it. Thus, an entrepreneur who enters such a market can always claim without fear of repercussions that he has spent all his money holdings and consumed the entire returns, and holds no resources to pay out to the investor. Further, since the investor and the entrepreneur lose track of each other at the end of the afternoon, no financial claims on the evening output, as well as on the afternoon output, can be written.

We assume that the morning output of entrepreneurs is fully pledgeable and that contracts between the entrepreneur and the investor can be made contingent on the early return of investment. Given the non-pledgeability described above, the payments must happen at the end of the morning before the afternoon market opens. A contract between an entrepreneur and an investor specifies the amount i of investment goods that the entrepreneur buys from the investor, which generates output with technology $f(i)$, and its payment – the entrepreneur pays out an amount z of the morning output (i.e. early return) and a fraction θ of his money holdings. Formally, z and θ must satisfy the following conditions. The first condition is the participation constraint of investors,

$$z + \theta\phi m = i,$$

where the L.H.S. represents the total payment of the entrepreneur and the R.H.S. is production costs of the investor. The entrepreneur makes an offer so that the investor is indifferent between producing or not. The amount ϕm represents the entrepreneur's real money holdings at the start of a given period. The second condition is *the liquidity constraint* given by

$$z \leq f(i).$$

The payment with output cannot exceed the early return that accrues during the morning, which is is the only part of the returns he can pledge. Note that since it is essential to conduct business using money in another market, i.e. consumption market, and holding money is costly under inflation, it is never optimal to pay for investment only with money, i.e. $z = 0$ cannot be part of a solution. Hence we ignore the constraint $z \geq 0$. Finally, the fraction θ has to satisfy

$$0 \leq \theta \leq 1. \tag{3}$$

Notice that we allow entrepreneurs to choose not to use money at this stage, i.e. θ can be set to zero. Using the participation constraint to substitute out $z = i - \theta\phi m$, we can write the liquidity constraint as

$$f(i) - i + \theta\phi m \geq 0. \tag{4}$$

Given values of i and $f(i)$, a larger amount of money pledged $\theta\phi m$ implies a smaller amount of output the entrepreneur has to pay out of his returns when the liquidity constraint (4) is binding. Thus, the use of money can mitigate the liquidity constraint. To summarize, entrepreneurs select i, θ that satisfy (3) and (4) in the investment market each period.

As already mentioned, the consumption good c is traded on a competitive market at a price p where traders are subject to frictions. Thus buyers need to offer money for their purchases of consumption, and face the budget constraint

$$pc \leq (1-\theta)m.$$

In equilibrium, the budget constraint turns out to be always binding: agents having a chance to spend their money holdings at this stage always want to exhaust them, since money is not needed for the rest of the period and carrying it into the future is costly under inflation. Observe that, because of the Inada condition, $u'(0) = \infty$, it is never optimal that $\theta = 1$. Hence we can ignore the constraint $\theta \leq 1$ in (3) above. The consumption and production on the afternoon market satisfies the respective first-order condition,

$$\begin{aligned} u'(c) &= (\rho + \phi)p \\ 1 &= \phi p, \end{aligned}$$

where ρ denotes the multiplier of the buyer's budget constraint. The consumption is determined so that the marginal utility equals the real market price ($= \phi p$) plus the cost of tightening the budget constraint ($= \rho p$), whereas the production is determined so that the marginal production cost equals the real market price.

We now describe the equilibrium conditions for the investment market. An entrepreneur selects i, θ to maximize the total expected returns of investment subject to (3) and (4). Denote by $\mu \geq 0$ the multiplier of the liquidity constraint (4). The first order condition for i is then given by

$$f'(i) + (1-\sigma)f'(i) = 1 + \mu(1 - f'(i)), \tag{5}$$

where the L.H.S. represents the total expected marginal returns of the investment, which is the sum of early and (expected) late returns, whereas the R.H.S. represents the marginal production costs plus the marginal cost of relaxing the liquidity constraint ($= \mu(1 - f'(\cdot))$). Comparing (5) to (1), one can see that if the liquidity constraint (4) is slack, $\mu = 0$, then the investment decision is made independently of the liquidity concerns and the outcome is efficient, $i = i^*$. If the liquidity constraint is binding $\mu > 0$, then the entrepreneur under-invests, $i < i^*$. Using the equilibrium conditions for the consumption market, which are reduced to $\phi m = c/(1-\theta)$, the complementary slackness condition for $\theta \geq 0$ in the liquidity constraint (4) can be written as

$$\mu \left[f(i) - i + \frac{\theta}{1-\theta} c \right] \;\; = \;\; 0. \tag{6}$$

Observe that the level of average productivity $f(i)/i$ relative to the average production costs ($= 1$) is going to play a crucial role in determining whether the liquidity constraint binds or not.

Similarly, denote by $\gamma \geq 0$ the multiplier for $\theta \geq 0$ in the constraint (3). Its complementary slackness condition is $\gamma\theta = 0$. The first order condition for θ is then given by

$$\frac{\gamma}{\phi m} + \mu = \frac{\sigma\rho}{\phi}$$

or

$$\frac{\gamma}{c} + \mu = \sigma \left(u'(c) - 1 \right). \tag{7}$$

To derive the latter expression (7), we use the equilibrium conditions for the consumption market, which are summarized by $\phi m = c/(1-\theta)$ and $\rho/\phi = u'(c) - 1$, and $\gamma\theta = 0$. The L.H.S. of (7) represents the marginal benefit of increasing an extra share of monetary payment θ, to relax the constraints (3) and (4). The R.H.S. represents the marginal opportunity cost of increasing θ, to reduce an extra unit of money holdings, measured by the net marginal expected utility of consumption, $u'(c) - 1$.

The choice of money holdings m – and thus, through the budget constraint $pc = (1-\theta)m$, the choice of consumption c – is governed by the Euler equation:

$$1 = \frac{\beta}{\pi} \left[(1-\theta)(\sigma u'(c) + 1 - \sigma) + (1+\mu)\theta \right].$$

It states that the marginal cost of obtaining an extra unit of money today ($= 1$) equals the discounted value – including the rate of price chance $\pi = \phi/\phi_{+1}$ – of its expected marginal benefit obtained tomorrow. Such marginal value has two components. First, an extra unit of money allows for further consumption: the entrepreneur can consume an extra unit during the afternoon as a buyer yielding $u'(\cdot)$ and during the night as a seller yielding 1. This return of money accrues from its role as a medium of exchange and is captured in the first term. Since a fraction θ of the money holdings has been spent before consumption occurs, this term is multiplied by $1-\theta$. Second, an extra unit of money reduces the need to pledge output to pay for investment. This return of money accrues from its role as enhancer of liquidity and is captured by the second term $(1+\mu)\,\theta$. The second role of money is absent when $\theta = 0$.

The Euler equation can be simplified to

$$\begin{aligned}
\frac{\pi}{\beta} - 1 &= \sigma(u'(c)-1) + \theta(\mu - \sigma(u'(c)-1)) \\
&= \sigma(u'(c)-1) - \frac{\gamma\theta}{c} \\
&= \sigma(u'(c)-1) \qquad (8)
\end{aligned}$$

where the second and third equality follow from the complementary slackness condition (7) and $\gamma\theta = 0$, respectively. The consumption c is determined solely by (8). Hence, the usual negative relationship between consumption and inflation holds.

Definition 1 *A steady state monetary equilibrium in our economy is a 5-tuple $(c, i, \mu, \theta, \gamma)$ satisfying equations (5), (6), (7), (8) and $\gamma\theta = 0$.*

Lemma 1 *If an equilibrium exists, then the consumption $c > 0$ is strictly decreasing in $\pi > \beta$ and satisfies $c \to c^*$ as $\pi \to \beta$.*

Notice that (7) implies that, given $u'(c) > 1$ in equilibrium, it is impossible to have $\mu = 0$ and $\theta > 0$, i.e. the liquidity constraint is not binding but a positive amount of money is pledged. In our model, the only role money can play in the investment market is to relax the liquidity constraint. Hence, in equilibrium the possible cases are: [1] the liquidity constraint is not binding, $\mu = 0$, and no money is pledged, $\theta = 0$; [2] the liquidity constraint is binding, $\mu > 0$, and a positive amount of money is pledged, $\theta > 0$; [3] the liquidity constraint is binding, $\mu > 0$, and no money is pledged, $\theta = 0$.

Consider case [1] first. With $\mu = 0$, (5) leads to $i = i^*$; with $\theta = 0$, (6) implies $f(i) \geq i$; finally, (7) determines γ. This is indeed the equilibrium behavior, when the average productivity at first best is greater or equal to the average production costs.

Proposition 1 *Suppose $f(i^*)/i^* \geq 1$. Then, a unique equilibrium exists for all $\pi > \beta$ in which the liquidity constraint is not binding, $\mu = 0$, and money is not pledged, $\theta = 0$, satisfying $i = i^*$.*

Consider case [2] next. With $\theta > 0$, it holds $\gamma = 0$ and $\mu = \frac{\pi}{\beta} - 1$ by (7) and (8); the latter expression can be used into (5) to obtain

$$f'(i) = \frac{\frac{\pi}{\beta}}{1 - \sigma + \frac{\pi}{\beta}},$$

where investment decreases with inflation; finally, with $\mu > 0$, (6) determines

$$\theta = \frac{i - f(i)}{c + i - f(i)}.$$

It can be seen here that, a level of investment will eventually be reached where $\theta = 0$ because the average productivity decreases with investment, due to the concavity of $f(i)$. This corresponds to case [3] for which (6) implies $i = f(i)$, (5) determines μ, and (7) determines γ. Indeed, a combination of cases [2] and [3] arises in equilibrium when the average productivity at first best is smaller than the average production costs.

Proposition 2 *Suppose $f(i^*)/i^* < 1$. Then, a unique equilibrium exists for all $\pi > \beta$ in which the liquidity constraint is binding, $\mu > 0$, satisfying $i \to i^*$ as $\pi \to \beta$. Further, there exists a unique $\hat{\pi} \in (\beta, \infty)$ such that $i = \hat{i} \in (0, i^*)$ at $\pi = \hat{\pi}$ and: $\theta > 0$ for $\pi \in (\beta, \hat{\pi})$ and $\theta = 0$ for $\pi \in [\hat{\pi}, \infty)$; $i \in (\hat{i}, i^*)$ is strictly decreasing in $\pi \in (\beta, \hat{\pi})$ and $i = \hat{i}$ for all $\pi \in [\hat{\pi}, \infty)$.*

The comparison of Proposition 1 and 2 reveals the role played by the liquidity constraint in the investment decisions of entrepreneurs. Proposition 1 shows that the constraint is never binding $\mu = 0$ for all $\pi > \beta$ if the average return of investment at first best is relatively high, $f(i^*)/i^* \geq 1$, while Proposition 2 shows that the constraint is binding $\mu > 0$ for all $\pi > \beta$ otherwise. In the former case, since the average pledgeable returns are high the liquidity constraint is irrelevant for the investment decision of entrepreneurs and equilibrium displays a dichotomous nature: the amount of entrepreneurs' investment is at the efficient level $i = i^*$ for all $\pi > \beta$, the investment market is insulated from monetary factors. In the latter case, the binding liquidity constraint implies costs to the investment of entrepreneurs and leads to under-investment $i < i^*$ for all $\pi > \beta$. Figure 2 illustrates the comparison of these two cases. In both cases the Friedman rule implements the efficient outcome both in terms of investment and consumption in our economy.

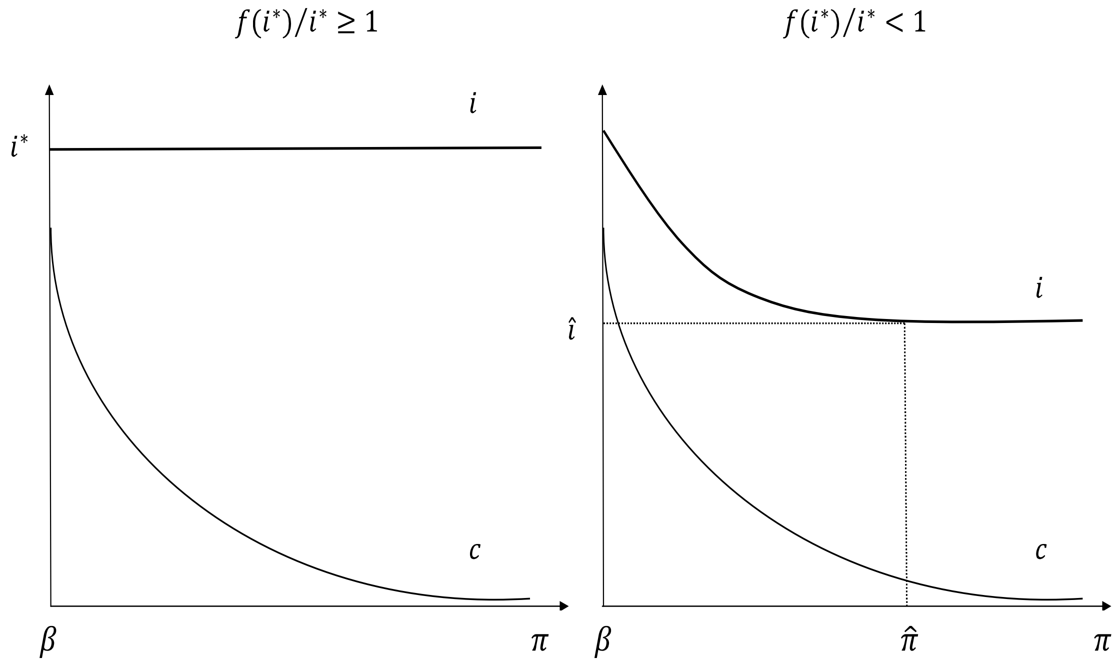

Figure 2: Equilibrium

Proposition 2 identifies the role of money in mitigating the liquidity constraint. When the average return of investment is relatively low, there is a relatively tight bound on the amount of output that can be pledged. This induces entrepreneurs to put some money up to relax the liquidity constraint. Indeed, a positive fraction of money holdings are used to pay for investment for low inflation rates, i.e., $\theta > 0$ for $\pi \in (\beta, \hat{\pi})$. Within this region, money provides liquidity services and the binding constraint can cause the monetary factor, and thus inflation, to distort decisions on the investment market: as inflation grows, the shadow cost of relaxing the liquidity constraint is increased, hence the investment level decreases with inflation, i.e., $\mu > 0$ is increasing and i is decreasing in $\pi \in (\beta, \hat{\pi})$. However, a lower investment level i implies a higher average return of the investment $f(i)/i$ which tends to relax the liquidity constraint. Thus, money becomes relatively less useful as a provider of liquidity services as the rate of inflation increases. For sufficiently high rates of inflation, money is not used anymore to pay for investment, i.e., $\theta = 0$ for $\pi \in [\hat{\pi}, \infty)$. Within this region, money plays no role as an enhancer of liquidity, and both $\mu > 0$ and i are constant for all $\pi \in [\hat{\pi}, \infty)$. Nevertheless, money still serves as a medium of exchange, thereby the consumption decreases as holding money becomes more costly, i.e., c decreases with $\pi \in (\beta, \infty)$, as stated in Lemma 1, irrespective of the productivity parameter.

2.3 Discussion and Extension

Restrictions on Payment Instruments For the sake of gaining a greater understanding of our results, it is instructive to compare the equilibrium investment level in our model and in settings which often appear in the CIA literature.

In our model, the net costs of relaxing the liquidity constraint, represented by $\mu(1 - f'(i))$, plays a critical role as far as under-investment is concerned. As we have seen, if $f(i^*)/i^* \geq 1$, then there is no under-investment; otherwise, $\mu(1 - f'(i)) =$

$$\left(\frac{\pi}{\beta} - 1\right)\left(\frac{1-\sigma}{1-\sigma+\frac{\pi}{\beta}}\right) \tag{9}$$

for $\pi \in (\beta, \hat{\pi})$, which increases in inflation, and $\mu(1 - f'(\hat{i})) =$

$$(2-\sigma)\, f'(\hat{i}) - 1, \tag{10}$$

where $\hat{i} = f(\hat{i})$, for $\pi \in [\hat{\pi}, \infty)$, which is independent of inflation.

Below, we consider two hypothetical settings featuring restrictions on what can be pledged at the investment stage: one assumes $z = 0$ and the other assumes $\theta = 0$. We leave the details in the Appendix and summarize the results in Figure 3. First, consider the case $z = 0$ (by assumption) whereby the investment returns cannot be pledged and the investment project has to be financed by cash only. This constitutes a version of CIA model *à* la Stockman (1981) in our environment. The liquidity constraint is now modified to

$$i \leq \phi m.$$

In this setup, both consumption and investment are financed entirely out of money holdings, and the choice of agents is on how much cash to allocate to either use. Therefore, the net cost of relaxing the liquidity constraint and of increasing consumption must be the same and equal to

$$\frac{\pi}{\beta} - 1, \tag{11}$$

which is the net cost of carrying money from one period to the next. Observe that (11) is higher than both (9) and (10). This is because being able to pledge the future returns can reduce the need to use money to relax the liquidity constraint. Thus, the amount of under-investment is relatively larger for all $\pi > \beta$ if financing investment is restricted to use money only. This result highlights the role of credit in mitigating inflationary distortions which take the form of under-investment.

Second, consider the case $\theta = 0$ (by assumption) whereby money cannot be pledged and the investment project has to be financed by its future returns. This setup is akin to a model with a CIA constraint on consumption only. The liquidity constraint is now modified to

$$i \leq f(i).$$

In this setup, if $f(i^*)/i^* \geq 1$, then the liquidity constraint is not binding and the first best level of investment can be achieved, i.e. $i = i^*$; otherwise, the constraint is binding and the investment satisfies $i = f(i)$, i.e. $i = \hat{i} < i^*$, which is independent of inflation and equal to the one derived in our (original) setup for $\pi \geq \hat{\pi}$. The cost of relaxing the liquidity constraint is given by (10) for all $\pi > \beta$. Thus, the amount of under-investments is relatively larger for $\pi < \hat{\pi}$ and remains the same for $\pi \geq \hat{\pi}$ if financing investment is restricted to use credit only. This result highlights the role of money in mitigating inflationary distortions for relatively row rates of inflation.

Proposition 3 *1. Suppose the future returns of investment cannot be pledged, i.e., credit is not possible. Then, the investment level is lower for all inflation rates, relative to our setup where both credit and cash can be used as payment instruments.*

2. Suppose cash cannot be used to pay for the investment costs, i.e., only credit is possible. Then, the investment level is lower for low inflation rates and equal for high inflation rates, relative to our setup where both credit and cash can be used as payment instruments.

To sum up, when inflation is low, the financial contract that allows the pledging of a mix of cash and future returns mitigates agents' under-investment relative to contracts restricting payment to either instrument alone. When inflation is high, money becomes useless as enhancer of liquidity, thereby the outcome is equivalent to the one ruling out cash payments. Further

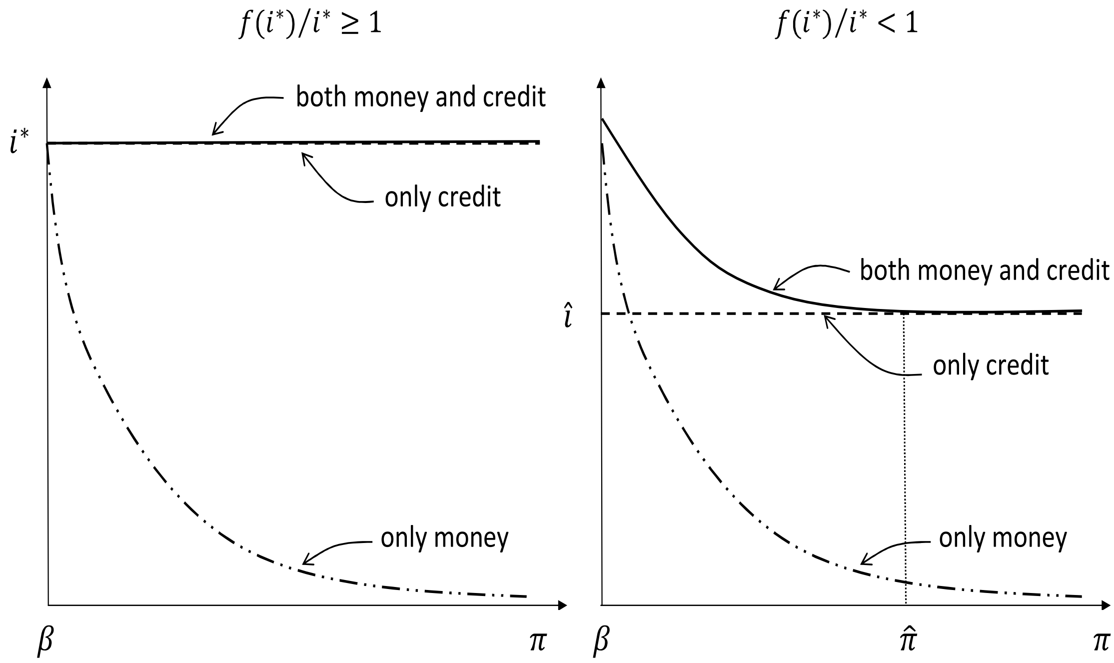

Figure 3: Comparison of investment level

implication of the results is that the welfare costs of inflation would be over-estimated if either one of these two payment instruments is not taken into account, especially at low inflation rates.

Feedback between Consumption and Investment We consider next an extension of the model which allows for a feedback between investment and consumption. Suppose that the investment good can be used to produce the consumption good, rather than yielding a late return.[2] This modified setup fits well the idea that part of the returns of investment cannot be pledged because of the informational and enforcement frictions in the consumption market. In addition, it has the flavor of the monetary search model with capital by Aruoba, Waller and Wright (2008), where a capital good can be used to produce a perishable consumption good.

Denote with $k(c^s, i)$ the effort cost of producing c^s units of the consumption good with i units of the investment good, measured in terms of utilities. Such a cost function can be obtained as a solution to the standard cost minimization problem subject to a neoclassical

[2] In general, we can postulate a late return function comprising both a direct late return as in the main body of the paper and a reduction in the production cost. This would lead to the same qualitative results.

production technology. This cost function satisfies $\frac{\partial k(c^s,i)}{\partial c^s} \equiv k_c > 0$, $\frac{\partial k(c^s,i)}{\partial i} \equiv k_i < 0$, $\frac{\partial^2 k(c^s,i)}{\partial c^{s2}} \equiv k_{cc} > 0$, $\frac{\partial^2 k(c^s,i)}{\partial i^2} \equiv k_{ii} > 0$. We assume that i is a normal good and thus $\frac{\partial^2 k(c^s,i)}{\partial c^s \partial i} \equiv k_{ci} < 0$.

Assuming $k_{cc}k_{ii} \geq (k_{ci})^2$, the planner's solution, denoted by c^{**}, i^{**}, can be characterized by the first order conditions

$$\begin{aligned} f'(i^{**}) - (1-\sigma)k_i(\varrho c^{**}, i^{**}) &= 1, \\ u'(c^{**}) &= k_c(\varrho c^{**}, i^{**}), \end{aligned}$$

where $\varrho \equiv \frac{\sigma}{1-\sigma}$ and the feasibility of the consumption good $c^s = \varrho c$ is already taken into account. The optimal solution exists and is unique (see the Appendix). Observe that the marginal late return $f'(\cdot)$ is now replaced by the marginal reduction of the production cost, $-k_i(\cdot) > 0$, and the marginal production cost of the consumption good is $k_c(\cdot)$, instead of 1. These two changes lead to the following modifications in the equilibrium conditions: the first order condition for the choice of i, (5), becomes

$$f'(i) - (1-\sigma)k_i(\varrho c, i) = 1 + \mu(1 - f'(i)); \tag{12}$$

the Euler equation (8) is now

$$\frac{\pi}{\beta} - 1 = \sigma\left(\frac{u'(c)}{k_c(\varrho c, i)} - 1\right); \tag{13}$$

the first order condition for the choice of θ, (7), should be modified accordingly.

Proposition 4 *Consider an alternative setup in which the investment good can be used to produce the consumption good, rather than yielding a late return.*

1. *When $f(i^{**})/i^{**} \geq 1$, a unique equilibrium exists for all $\pi > \beta$ where the liquidity constraint is not binding, $\mu = 0$, and money is not pledged, $\theta = 0$. In this case, both consumption and investment are decreasing in all $\pi \in (\beta, \infty)$.*

2. *When $f(i^{**})/i^{**} < 1$, a unique equilibrium exists for all $\pi > \beta$ where the liquidity constraint is binding, $\mu > 0$, and consumption is decreasing in all $\pi \in (\beta, \infty)$. Further, there exists a unique $\tilde{\pi} \in (\beta, \infty)$ such that $\theta > 0$ and investment is decreasing for $\pi \in (\beta, \tilde{\pi})$, while $\theta = 0$ and investment is constant for $\pi \in [\tilde{\pi}, \infty)$.*

In case 1, since agents are not liquidity constrained to finance investment, the behavior of investment is governed by its productive concerns: lower consumption due to inflation must be accompanied by lower investment, provided that the investment good is a normal good, i.e. provided $k_{ic} < 0$. The role of the investment to reduce production costs becomes smaller as money becomes more costly to hold. This effect was absent in our benchmark economy. The same effect is still at work in case 2, where the liquidity constraint is binding, but is accompanied by the original effect of liquidity we identified in Proposition 2. The behavior of investment is essentially the same as before: when inflation is relatively low both money and early returns are used to finance investment, in which case the combined effects of production and liquidity, which work in the same direction, make investment decreasing in inflation; when inflation is relatively high only early returns are used, in which case the behavior of investment is dictated by its liquidity concerns, thereby the binding liquidity constraint determines the equilibrium investment level which is insensitive to inflation.

Empirical Evidence There is by now ample evidence of an empirically strong positive association between investment and internal funds, starting with a paper by Fazzari, Hubbard and Pedersen (1988). Internal funds need not be held in the form of non-interest bearing cash, it may be countered. However, recent micro-based empirical evidence by Denis and Sibilkov (2009) shows that firms – especially financially constrained ones – do hold a substantial amount of their funds in cash and they do so in order to finance investment. Using a sample of 74,347 firm-year observations between 1985 and 2006 for the US, they find that median cash holdings of financially constrained firms are around 10% of their assets and their net investment is positively and significantly related to cash holdings, with the figures being smaller for unconstrained firms. In addition, several papers, e.g. Mulligan (1997), contain a firm-level evidence of a negative effect of inflation on the demand for cash by firms.

As regards the macro evidence, Barro (1996) – using data for 100 countries over the period 1960-1990 – found that an increase in average inflation would significantly lower the investment to GDP ratio. Madsen (2003), using panel data for the OECD countries, found that investment in non-residential buildings, structures, machinery and equipment is strongly negatively related to inflation. Being investment highly correlated with financial market activity, some papers, notably Boyd, Levine and Smith (2001), have been looking at the relationship between steady state inflation and financial market activity. Using data on inflation, banking sector activity, equity market size, equity market liquidity, and the rates of return on stocks for 100 countries over the period 1960–1995, averaged over the entire period, so as to concentrate on the long-run, steady state relationship between inflation and financial sector conditions, they consistently find a negative and significant relationship of steady state inflation on financial market activity, with the effect dying out at high inflation rates. They also find evidence that these effects are stronger for less developed countries (see also the book edited by Demirguç-Kunt and Levine (2001)). This is consistent with our model, which predicts a differential impact of inflation on investment according to the stage of development of a country as represented both by its market frictions and its level of technological sophistication.

3 Conclusion

We have presented a simple framework using a standard monetary search approach that allows us to study the issue of liquidity and its effect on investment. The gist of the paper is that when firms are credit constrained they use internal liquid funds in the form of cash holdings to finance investment. Through this channel monetary policy affects firms decisions and steady state inflation depresses investment, up to a certain inflation threshold after which the effect disappears.

To achieve our objective, we have paid particular attention to the relationship between money as a medium of spot trade and a medium of trade over time. An interesting extension is to add another credit instrument/medium of exchange, e.g., interest-bearing liquid asset, along the lines of Wright (2008). This additional ingredient could capture a real-life phenomenon that firms can hold liquidity in the form of many different types of liquid assets. Another direction would be to test empirically one of the implications of our model: even when markets do not function smoothly and contracts are poorly enforced, a sufficient level of technological sophistication of the productive sector/industry may allow the economy to avoid major disruptions to investment; when the economy is technologically less developed, instead, inflation could have a sizable adverse effect on investment.

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4 Appendix

In this Appendix, we provide a formal derivation of the equilibrium conditions stated in the main text and the proof of Proposition 1 – 4. We derive the value function only for entrepreneurs (the active group), and not for investors (the passive group). Since there is no reason for investors to carry money into the future, one can assume without loss of generality that they will spend it all in the evening of the same period and they will not carry any money from one period to the next.

To derive the equilibrium conditions, we work backward and start with the evening market. During the evening, agents trade, consume and produce output. At the start of any given evening, the expected value of an entrepreneur who holds $\hat{m}$ money and enters the evening market, denoted by $W(\hat{m})$, satisfies

$$\begin{aligned} W(\hat{m}) \quad &= \quad \max_{x,e,m_{+1}\geq 0} x - e + \beta V(m_{+1}) \\ &\quad\quad \text{s.t.} \quad x - e = \phi(\hat{m} - m_{+1}) + \tau \end{aligned}$$

where $V(m_{+1})$ denotes the expected value of entering into the next morning market with holdings m_{+1} of money. The nominal price in the evening market is normalized to 1, and so ϕ represents the relative price of money. Given these prices, the initial money holding $\hat{m}$ and the government tax or transfer τ, the agent chooses an amount of consumption x, effort e and the future money holdings m_{+1}. Note that the initial money holding $\hat{m}$ at the start of a given evening depends on the agent's activities during the morning and afternoon of the same period. If an entrepreneur has started the morning with m money, paid θ money to the investor, and sold c^s (or bought c) units on the afternoon market at a price p, then his initial money holding at the start of the evening is given by $\hat{m} = (1-\theta)m + pc^s$ (or $\hat{m} = (1-\theta)m - pc$).

Substituting out the term $x - e$ in the value function using the constraint, we obtain the first order condition

$$\beta V'(m_{+1}) = \phi. \tag{14}$$

Observe that m_{+1} is determined independently of $\hat{m}$ (and of m), and hence all entrepreneurs hold the same amount of money at the beginning of any given morning market.

Entrepreneurs either buy and consume goods, or produce and sell them on the market during the afternoon. The expected value of an entrepreneur who holds i investment goods and $(1-\theta)m$ money, and enters the afternoon market, denoted by $Z(i,(1-\theta)m)$, satisfies

$$\begin{aligned} Z(i,(1-\theta)m) &= \sigma \left\{ \begin{array}{c} \max_{c \geq 0} \; u(c) + W((1-\theta)m - pc) \\ \text{s.t. } pc \leq (1-\theta)m \end{array} \right\} \\ &\quad + (1-\sigma) \left\{ \max_{c^s \geq 0} \; f(i) - c^s + W((1-\theta)m + pc^s) \right\}. \end{aligned}$$

If the entrepreneur turns out to be a buyer, which happens with probability σ, then he can buy and consume the consumption goods c up to his money holdings $(1-\theta)m$ at the market price p. He then carries $(1-\theta)m - pc$ money to the evening. $W((1-\theta)m - pc)$ is his continuation value specified before. If the entrepreneur turns out to be a seller, which happens with probability $1-\sigma$, then he can produce an amount of the consumption goods, denoted by c^s, with unit marginal costs and sell it at p. The seller who has invested an amount i in the morning obtains and consumes the late returns of the investment, $f(i)$, during the afternoon. The seller's continuation value is given by $W((1-\theta)m + pc^s)$. Using the envelope conditions, $\partial W(\cdot)/\partial c = -\phi p$ for the buyer and $\partial W(\cdot)/\partial c^s = \phi p$ for the seller, we derive the first order conditions, $u'(c) = (\rho + \phi)p$ and $1 = \phi p$, as given in the main text. The complementary slackness condition for the budget constraint is

$$\rho((1-\theta)m - pc) = 0. \tag{15}$$

At the start of each period, each entrepreneur offers a randomly matched investor the contract described in the text, which specifies a payment z, θ out of their resources in exchange for an amount of investment goods i. The repayment happens at the end of the morning. An entrepreneur who holds m money at the start of any given morning has the expected value, denoted by $V(m)$, satisfying

$$V(m) = \max_{i,z,\theta \geq 0} \left[f(i) - z + Z(i,(1-\theta)m) \right]$$

subject to the participation constraint $z + \phi\theta m = i$, the liquidity constraint $z \leq f(i)$, and (3). After paying out z output and $\theta\phi m$ money in real terms to the investor, the entrepreneur can consume the remaining $f(i) - z$ morning output (i.e., an early return of the investment net of the output-payment) and carry the remaining $(1-\theta)m$ money to the afternoon. $Z(\cdot)$ is the continuation value described before. Solving the participation constraint for $z = i - \phi\theta m$ and applying this solution to the value function, we can reduce the programme to the following form:

$$V(m) = \max_{i,\theta \geq 0} \left[f(i) - i + \theta\phi m + Z(i,(1-\theta)m) \right]$$

subject to (3) and (4). Using this expression and the envelope conditions, $\partial Z(\cdot)/\partial i = (1-\sigma)f'(i)$ and $\partial Z(\cdot)/\partial\theta = -(\sigma\rho+\phi)m$, we can derive the first-order conditions for the morning market as in the main text. The complementary slackness conditions are $\gamma\theta = 0$ and

$$\mu\left[f(i) - i + \theta\phi m\right] = 0. \tag{16}$$

Now, the envelope condition for m is

$$V'(m) = \phi\left[(1-\theta)\left(\frac{\sigma\rho}{\phi}+1\right) + (1+\mu)\,\theta\right].$$

Plugging this expression into (14) with an updating and rearranging it using the first-order conditions for the afternoon market, which are reduced to $\rho/\phi = u'(c) - 1$, we obtain the Euler equation for money holdings m:

$$\phi = \beta\phi_{+}\left[\ (1-\theta)\left(\sigma u'(c) + 1 - \sigma\right) + (1+\mu)\,\theta\ \right].$$

The last expression leads to the one presented in the main text (which is further simplified to (8)) by using the money market clearing condition

$$\frac{\phi_{+1}}{\phi} = \frac{1}{\pi}.$$

The other market clearing conditions are

$$\sigma c = (1-\sigma)c^{s}$$

in the afternoon, and the one in the evening that can be ignored by virtue of Walras Law.

The above analysis provides a complete description of the equilibrium conditions in our economy. We now prove the binding budget constraint of buyers. Note that we have not used this fact above, while it will be used to establish an equilibrium.

Lemma 2 *For $\pi > \beta$, the budget constraint of buyers must be binding, i.e. $(1-\theta)m = pc$.*

Proof of Lemma 2. Suppose $\rho = 0$. Then, $u'(c) = (\rho+\phi)p$ and $1 = \phi p$ imply that $u'(c) - 1 = \rho p$. This, however, contradicts (8) for $\pi > \beta$, hence if a solution exists for $\pi > \beta$ then we must have $\rho > 0$, leading to $(1-\theta)m = pc$ by (15). This completes the proof of Lemma 2.■

With the binding budget constraint $(1-\theta)m = pc$, the complementary slackness condition (16) becomes (6), and the first-order condition for θ becomes (7), as stated in the main text. To identify a steady state monetary equilibrium is now reduced to identify i, θ, μ, c that satisfies equations (5) to (8), and γ that satisfies $\gamma\theta = 0$. Given this observation, we now provide the proof of the existence and uniqueness of equilibrium.

Proof of Proposition 1.

In what follows we use the following properties: as $f(\cdot)$ is a strictly concave function and satisfies $f(0) = 0$, it follows that

$$\frac{f(i)}{i} > f'(i) \tag{17}$$

for all $i \in (0, \infty)$ and $f(i)/i$ is strictly decreasing in $i \in (0, \infty)$. Given $f(i^*)/i^* \geq 1$, the proof of Proposition 1 proceeds with the following steps: Step 1 shows $\mu, \theta > 0$ cannot be a solution; Step 2 shows $\mu > 0$, $\theta = 0$ cannot be a solution. By Step 1 and 2, since $\mu = 0, \theta > 0$ are not possible, the only possible case is $\mu = \theta = 0$, implying $\gamma > 0$ by $\gamma\theta = 0$. In this case, (5) with $\mu = 0$ identifies a unique solution $i = i^*$, which satisfies (6) and is independent of π. With $c \in (0, c^*)$ satisfying (8), a unique $\gamma > 0$ is identified by (7). This solution in turn satisfies $\gamma\theta = 0$ and (5) - (7) and so it is a unique equilibrium.

Step 1 If $f(i^*)/i^* \geq 1$, then $\mu, \theta > 0$ cannot be a solution.

Proof of Step 1. Suppose $\mu > 0$ and $\theta > 0$. $\theta > 0$ implies $\gamma = 0$ by $\gamma\theta = 0$. Applying $\gamma = 0$ to (7) and using (5), (7), (8), we get

$$\frac{\pi}{\beta} - 1 = \frac{(2-\sigma)f'(i) - 1}{1 - f'(i)}.$$

The R.H.S. of this equation is strictly decreasing in $i \in (f'^{-1}(1), i^*)$. This equation has a unique solution $i = i(\pi)$, which is strictly decreasing in $\pi > \beta$ and satisfies $i(\pi) \to i^* \equiv f'^{-1}(1/(2-\sigma))$ as $\pi \to \beta$ and $i(\pi) \to f'^{-1}(1)$ as $\pi \to \infty$. This further implies $i < i^*$ for all $\pi > \beta$. As $f(i)/i$ is strictly decreasing in $i \in (0, \infty)$, we must have

$$\frac{f(i)}{i} > \frac{f(i^*)}{i^*} \geq 1$$

for all $\pi > \beta$. However, (6) and $\mu > 0$ require

$$f(i) - i = -\frac{\theta}{1-\theta}c$$

implying $f(i)/i > 1$ contradicts $\theta > 0$. This completes the proof of Step 1.

Step 2 If $f(i^*)/i^* \geq 1$, then $\mu > 0$, $\theta = 0$ cannot be a solution.

Proof of Step 2. Suppose $\mu > 0$ and $\theta = 0$. $\mu > 0$ and $\theta = 0$ imply $f(i) = i$ by (6). For $f(i^*)/i^* \geq 1$ this is possible only when $f(i^*) = i^*$ and $i = i^*$. However, this contradicts $\mu > 0$ because applying $i = i^*$ to (5) yields

$$\mu = \frac{(2-\sigma)f'(i^*) - 1}{1 - f'(i^*)} = 0.$$

This completes the proof of Step 2. ■

Proof of Proposition 2.

Given $f(i^*)/i^* < 1$, the proof proceeds with similar steps as before. Step 1 shows $\mu = \theta = 0$ cannot be a solution. As $\mu = 0, \theta > 0$ is not possible, this implies the only possible cases are either $\mu, \theta > 0$ or $\mu > 0$, $\theta = 0$. Using $\gamma\theta = 0$ and (5) - (8), Step 2 then shows that there exists a unique $\hat{\pi} \in (\beta, \infty)$ such that $\theta > 0$ for $\pi \in (\beta, \hat{\pi})$ and $\theta = 0$ for $\pi \in [\hat{\pi}, \infty)$. In the former region, we have $\gamma = 0$, (5), (7), (8) identify a unique $i = i(\pi)$, which is strictly decreasing in $\pi \in (\beta, \hat{\pi})$, and (6) identifies a unique $\theta \in (0, 1)$. In the latter region, we have $\gamma \geq 0$ satisfying (5), (7), (8), and (6) identifies a unique i, which is independent of $\pi \in [\hat{\pi}, \infty)$. With $c \in (0, c^*)$ satisfying (8) and $\mu > 0$ satisfying (5), this solution in turn satisfies $\gamma\theta = 0$, (5) - (8) and so it is a unique equilibrium.

Step 1 If $f(i^*)/i^* < 1$, then $\mu = \theta = 0$ cannot be a solution.

Proof of Step 1. Suppose $\mu = \theta = 0$. $\mu = 0$ implies $i = i^*$ by (5). However, (6) requires that $f(i) \geq i$ hence $f(i^*) \geq i^*$, which contradicts $f(i^*)/i^* < 1$. This completes the proof of Step 1.

Step 2 If $f(i^*)/i^* < 1$, then there exists a unique $\hat{\pi} \in (\beta, \infty)$ such that $\theta > 0$ for $\pi \in (\beta, \hat{\pi})$ and $\theta = 0$ for $\pi \in [\hat{\pi}, \infty)$.

Proof of Step 2. Suppose $\mu > 0$ and $\theta > 0$. Then as shown in the Step 1 in the proof of Proposition 1, there exists a unique solution $i = i(\pi)$ $(< i^*)$ to (5), (7), (8), which is strictly decreasing in $\pi > \beta$ and satisfies $i(\pi) \to i^*$ as $\pi \to \beta$ and $i(\pi) \to f'^{-1}(1)$ as $\pi \to \infty$. Observe (17) implies at $i = f'^{-1}(1)$ we have

$$\frac{f\left(f'^{-1}(1)\right)}{f'^{-1}(1)} > 1 = f'\left(f'^{-1}(1)\right).$$

As $f(i)/i$ is strictly decreasing in $i \in (0, \infty)$, $i = i(\pi) \in (f'^{-1}(1), i^*)$ is strictly decreasing in $\pi \in (\beta, \infty)$ and $f(i^*)/i^* < 1$, this implies that there exists a unique $\hat{\pi} \in (\beta, \infty)$ such that $\hat{i} = i(\hat{\pi}) \in (f'^{-1}(1), i^*)$ and

$$\frac{f(\hat{i})}{\hat{i}} = 1.$$

This further implies $f(i) - i < 0$ for $\pi \in (\beta, \hat{\pi})$ and $f(i) - i \geq 0$ for $\pi \in [\hat{\pi}, \infty)$. Therefore, given $c > 0$ satisfying (8), it follows that (6) with $\mu > 0$ identifies a unique $\theta \in (0, 1)$ for $\pi \in (\beta, \hat{\pi})$. For $\pi \in [\hat{\pi}, \infty)$ the only remaining possibility is the case $\mu > 0$ and $\theta = 0$.

Suppose now that $\mu > 0$ and $\theta = 0$. Then, (6) determines a unique $i = \hat{i}$ $(< i^*)$ which is independent of π. On the other hand, (5), (7) and (8) imply

$$\frac{\gamma}{c} = \frac{\pi}{\beta} - 1 - \frac{(2 - \sigma)f'(\hat{i}) - 1}{1 - f'(\hat{i})}.$$

This expression shows, given $c > 0$ satisfying (8), we must have $\gamma > 0$, implying $\theta = 0$, if and only if $\pi \in (\hat{\pi}, \infty)$. At $\pi = \hat{\pi}$, it holds that $\gamma = \theta = 0$. This completes the proof of Step 2. ■

Proof of Proposition 3.

$\odot$ **Money only (No credit as a payment instrument):** The value function of an entrepreneur with m money in the morning is now modified to

$$V(m) = \max_{i \geq 0} \{f(i) + Z(i, m - i/\phi)\}$$

subject to the liquidity constraint $i \leq \phi m$, where $Z(i, m - i/\phi)$ represents the value of the entrepreneur in the afternoon with i investment goods and $m - i/\phi$ money, given by

$$\begin{aligned} Z(i, m - i/\phi) &= \sigma \left\{ \begin{array}{c} \max_{c \geq 0} \; u(c) + W(m - i/\phi - pc) \\ \text{s.t. } pc \leq m - i/\phi \end{array} \right\} \\ &\quad + (1 - \sigma) \left\{ \max_{c^s \geq 0} \; f(i) - c^s + W(m - i/\phi + pc^s) \right\}. \end{aligned}$$

The first order condition for consumption,

$$u'(c) = \frac{\rho}{\phi} + 1 \tag{18}$$

remains the same as before, while the first order condition for investment and the Euler equation are given respectively by

$$\begin{aligned} (2 - \sigma) f'(i) &= 1 + \mu + \frac{\sigma \rho}{\phi}, & (19) \\ \frac{\pi}{\beta} &= \sigma(u'(c) - 1) + 1 + \mu. & (20) \end{aligned}$$

The complementary slackness conditions are

$$\begin{aligned} \rho(\phi m - c - i) &= 0 & (21) \\ \mu(\phi m - i) &= 0. & (22) \end{aligned}$$

Equilibrium implies $\mu = 0$ and $\rho > 0$, for $\pi > \beta$, since if $\rho = 0$ then (18) implies $u'(c) = 1$, a contradiction to (20) and $\pi/\beta > 1$. Hence, $\phi m = c + i$ by (21), which further implies we must have $\mu = 0$ in (22) as $c = 0$ cannot be a solution. With $\mu = 0$, (20) implies the solution $c \in (0, \infty)$ exists and is unique for all $\pi \in (\beta, \infty)$, satisfying $c \to \infty$ as $\pi \to \beta$ and $c \to 0$ as $\pi \to \infty$, i.e., following the same path as before (with credit). Using (18), (19), (20),

$$f'(i) = \frac{\pi/\beta}{2 - \sigma}$$

which implies the solution $i \in (0, i^*)$ exists and is unique, satisfying $i \to i^*$ as $\pi \to \beta$ and $i \to 0$ as $\pi \to \infty$.

For the sake comparison, denote by i^m the equilibrium investment with only money as a payment instrument, while keeping i to represent the original investment with money and credit. If $f(i^*)/i^* \geq 1$ then $i = i^* = f^{-1\prime}(1/(2 - \sigma)) > f^{-1\prime}(\pi/\beta/(2 - \sigma)) = i^m$, thereby $i > i^m$ for all $\pi > \beta$. If $f(i^*)/i^* < 1$ then

$$f'(i) = \frac{\pi/\beta}{1 - \sigma + \pi/\beta} < \frac{\pi/\beta}{2 - \sigma} = f'(i^m)$$

and so $i > i^m$ for $\pi \in (\beta, \hat{\pi})$. For $\pi \in [\hat{\pi}, \infty)$ we have

$$i = f^{-1\prime}\left(\frac{\pi/\beta}{1-\sigma+\pi/\beta}\right) > f^{-1\prime}\left(\frac{\pi/\beta}{2-\sigma}\right) = i^m.$$

Therefore, $i > i^m$ for all $\pi \in (\beta, \infty)$. ■

⊙ **Credit only (No money as a payment instrument):** The value function of an agent with m money in the morning is now modified to

$$V(m) = \max_{i \geq 0} \{f(i) - i + Z(i, m)\}$$

subject to the liquidity constraint $i \leq f(i)$, where $Z(i, m)$ is defined as before. The first order conditions for consumption and investment remain the same as before. The Euler equation is given by $\frac{\pi}{\beta} = \sigma(u'(c) - 1) + 1$, and the complementary slackness conditions are $\rho(\phi m - c) = 0$ and

$$\mu(f(i) - i) = 0. \tag{23}$$

The usual procedure implies that we must have $\rho > 0$ for $\pi > \beta$, and that the solution $c \in (0, \infty)$ exists and unique. As before, whether or not the liquidity constraint is binding depends on $f(i^*)/i^*$. Observe that the lagrange multiplier $\mu \geq 0$ given by (5) is strictly decreasing in $i \leq i^*$, satisfying $\mu = 0$ when $i = i^*$ and $\mu = \infty$ when $i = f^{-1\prime}(1)$. Consider first the case $f(i^*)/i^* \geq 1$. In this case, $\mu > 0$ cannot be a solution because if that is the case, we must have $f(i)/i = 1$ by (23), which contradicts that $f(i)/i$ is strictly decreasing in $i \leq i^*$. Therefore, if $f(i^*)/i^* \geq 1$ we have $\mu = 0$ and $i = i^*$ for all $\pi > \beta$. Consider next the case $f(i^*)/i^* < 1$. In this case, $\mu = 0$ cannot be a solution because if that is the case we must have $i = i^*$ by (5), which contradicts $f(i)/i \geq 1$ in (23) and $f(i^*)/i^* < 1$. Therefore, if $f(i^*)/i^* < 1$ we have $\mu > 0$ and $i = \hat{\hat{i}} < i^*$ for all $\pi > \beta$. ■

Proof of Proposition 4.

In the alternative setup where the investment good can be used for producing the consumption good, the value function of entrepreneurs during the afternoon is modified to

$$\begin{aligned} Z(i, (1-\theta)m) &= \sigma \left\{ \begin{array}{c} \max_{c \geq 0} \; u(c) + W((1-\theta)m - pc) \\ \text{s.t. } pc \leq (1-\theta)m \end{array} \right\} \\ &\quad + (1-\sigma)\left\{ \max_{c^s \geq 0} \; -k(c^s, i) + W((1-\theta)m + pc^s) \right\}. \end{aligned}$$

Accordingly, the envelope condition is now changed to $\partial Z(\cdot)/\partial i = -(1-\delta)k_i(\cdot)$, which leads to the first order condition for i, (12), applying the market clearing condition $c^s = \varrho c$ as in the main text, and the seller's first order condition for c^s to

$$k_c(c^s, i) = \phi p.$$

Combined with the buyer's first order condition for c, $u'(c) = (\rho + \phi)p$, this leads to

$$\frac{\rho}{\phi} = \frac{u'(c)}{k_c(c^s, i)} - 1,$$

which further changes the first order condition for θ, (7), to

$$\frac{\gamma}{c} + \mu = \sigma \left(\frac{u'(c)}{k_c(c^s, i)} - 1 \right). \tag{24}$$

Plugging these expressions into the envelope condition for m and (14), with $c^s = \varrho c$ and an updating and rearrangement, one can obtain the Euler equation for money holding (13) as in the main text. These are the only modifications. The equilibrium conditions in this alternative setup are summarized by the first order conditions for i, (12), and for θ, (24), the Euler equation, (13), and the complementary slackness conditions, (6) and $\gamma\theta = 0$. Before describing the equilibrium, we shall start with the planner's solution.

⊙ **Existence and uniqueness of the planner's solution $c^{**}, i^{**} \in (0, \infty)$:** The solution must satisfy the first order conditions

$$\begin{aligned} \Phi_p(c, i) &\equiv f'(i) - (1-\sigma)k_i(\varrho c, i) - 1 = 0, \\ \Psi_p(c, i) &\equiv u'(c) - k_c(\varrho c, i) = 0. \end{aligned}$$

Observe

$$\frac{\partial \Phi_p}{\partial c} = -(1-\sigma)\varrho k_{ic} > 0, \ \frac{\partial \Phi_p}{\partial i} = f''(i) - (1-\sigma)k_{ii} < 0,$$
$$\frac{\partial \Psi_p}{\partial c} = u''(c) - \varrho k_{cc} < 0, \ \frac{\partial \Psi_p}{\partial i} = -k_{ci} > 0.$$

The former implicit equation defines an implicit function $i_i = i_i(c) \in (f'^{-1}(1), \infty)$ for $c \in [0, \infty)$ that satisfies

$$\frac{di_i(c)}{dc} = \frac{(1-\sigma)\varrho k_{ic}}{f''(i) - (1-\sigma)k_{ii}} > 0,$$

and $i_i(c) \to \bar{i}_i \in (f'^{-1}(1), \infty)$ as $c \to \infty$ and $i_i(0) = \underline{i}_i \in (f'^{-1}(1), \bar{i}_i) > 0$. Similarly, the latter defines a function $i_c = i_c(c) \in [0, \infty)$ for $c \in (0, \infty)$ that satisfies

$$\frac{di_c(c)}{dc} = \frac{u''(c) - \varrho k_{cc}}{k_{ci}} > 0,$$

$i_c(\underline{c}) = 0$, where $\underline{c} \in (0, \infty)$, and $i_c(c) \to \infty$ as $c \to \bar{c} \in (\underline{c}, \infty)$. The solution is then identified by finding a fixed point of

$$i_c(c) = i_i(c).$$

Notice that: (i) $i_c(\underline{c}) = 0 < \underline{i}_i < i_i(\underline{c})$; (ii) $i_c(c) \to \infty > \bar{i}_i > i_i(c)$ as $c \to \bar{c}$; (iii) $\frac{di_c(c)}{dc} - \frac{di_i(c)}{dc} =$

$$\frac{u''(c)\left\{f''(i) - (1-\sigma)k_{ii}\right\} - \varrho k_{cc} f''(i) + (1-\sigma)\varrho\left(k_{ii}k_{cc} - k_{ic}^2\right)}{k_{ci}\left\{f''(i) - (1-\sigma)k_{ii}\right\}} > 0,$$

given our assumption $k_{ii}k_{cc} - k_{ic}^2 > 0$. Therefore, there exists a unique fixed point that satisfies $c = c^{**} \in (\underline{c}, \bar{c}) \subseteq (0, \infty)$ and $i = i^{**} \in (\underline{i}_i, \bar{i}_i) \subseteq (f'^{-1}(1), \infty)$. ■

We now prove the existence and uniqueness of the equilibrium, and provide the comparative statics result.

⊙ **For** $f(i^{**})/i^{**} \geq 1$**:** The proof of $\mu = \theta = 0$ for $f(i^{**})/i^{**} \geq 1$ is identical to the one presented in the proof of Proposition 1 and we shall not repeat it. The equilibrium c, i in this case must satisfy $\Phi_p(c, i) = 0$ (defined above) and

$$\Psi_n(c, i; \pi) \quad \equiv \quad \sigma \left(\frac{u'(c)}{k_c(\varrho c, i)} - 1 \right) - \frac{\pi}{\beta} + 1 = 0$$

which is constructed by (13). Observe

$$\frac{\partial \Psi_n}{\partial c} = \sigma \frac{u''(c)k_c(\cdot) - \varrho u'(c)k_{cc}}{k_c(\cdot)^2} < 0, \quad \frac{\partial \Psi_n}{\partial i} = -\frac{\sigma u'(c)k_{ci}}{k_c(\cdot)^2} > 0.$$

The implicit equation $\Psi_n(c, i; \pi) = 0$ defines an implicit function $i_{c'} = i_{c'}(c; \pi) \in [f'^{-1}(1), \infty)$ for $c \in (0, \infty)$ and $\pi \in (\beta, \infty)$ that satisfies

$$\frac{di_{c'}(c; \pi)}{dc} = \frac{u''(c)k_c(\cdot) - \varrho u'(c)k_{cc}}{u'(c)k_{ci}} > 0,$$

$i_{c'}(c^{**}) > i^{**}$ (as $\pi > \beta$) and $i_{c'}(c_\pi) = f'^{-1}(1)$, where $c_\pi = c_\pi(\pi) \in (0, c^{**})$ for all $\pi \in (\beta, \infty)$.

The fixed point condition is then given by

$$i_{c'}(c; \pi) = i_i(c)$$

where $i_i(c) \in (\underline{i}_i, \bar{i}_i)$ is defined before and satisfies $\Phi_p(c, i_i(c)) = 0$. Notice that: (i) $i_{c'}(c_\pi) = f'^{-1}(1) < \underline{i}_i < i_i(c_\pi)$; (ii) $i_{c'}(c^{**}) > i^{**} = i_i(c^{**})$; (iii) $\frac{di_{c'}(c;\pi)}{dc} - \frac{di_i(c)}{dc} =$

$$\frac{u''(c)k_c(\cdot)\left\{f''(i) - (1-\sigma)k_{ii}\right\} - \varrho u'(c)k_{cc}f''(i) + (1-\sigma)\varrho u'(c)\left(k_{ii}k_{cc} - k_{ic}^2\right)}{u'(c)k_{ci}\left\{f''(i) - (1-\sigma)k_{ii}\right\}} > 0.$$

Therefore, there exists a unique fixed point that satisfies $c \in (c_\pi, c^{**})$ and $i \in (\underline{i}_i, i^{**}) \subseteq (f'^{-1}(1), i^{**})$ for all $\pi \in (\beta, \infty)$.

We now examine the comparative statics. Consider first c. To reflect the dependence of the equilibrium c on π, denote by $c = c(\pi)$ and write the fixed point condition as $i_{c'}(c(\pi); \pi) = i_i(c(\pi))$. Total differentiation yields

$$\frac{dc(\pi)}{d\pi} = -\frac{\frac{\partial i_{c'}(c;\pi)}{\partial \pi}}{\frac{di_{c'}(c;\pi)}{dc} - \frac{di_i(c)}{dc}} < 0$$

as both the numerator and the denominator are positive. To see the comparative statics on $i = i(\pi)$, observe that

$$\frac{di(\pi)}{d\pi} = -\frac{\det \begin{bmatrix} \frac{\partial \Phi_p}{\partial \pi} & \frac{\partial \Phi_p}{\partial c} \\ \frac{\partial \Psi_n}{\partial \pi} & \frac{\partial \Psi_n}{\partial c} \end{bmatrix}}{\det \begin{bmatrix} \frac{\partial \Phi_p}{\partial i} & \frac{\partial \Phi_p}{\partial c} \\ \frac{\partial \Psi_n}{\partial i} & \frac{\partial \Psi_n}{\partial c} \end{bmatrix}} = \frac{\frac{\partial \Phi_p}{\partial c}\frac{\partial \Psi_n}{\partial \pi}}{\frac{\partial \Phi_p}{\partial i}\frac{\partial \Psi_n}{\partial c} - \frac{\partial \Phi_p}{\partial c}\frac{\partial \Psi_n}{\partial i}}$$

where the second equality follows from $\partial\Phi/\partial\pi = 0$. The numerator of the above expression is negative, while the denominator can be rearranged as

$$\frac{\partial\Phi_p}{\partial i}\frac{\partial\Psi_n}{\partial c} - \frac{\partial\Phi_p}{\partial c}\frac{\partial\Psi_n}{\partial i} = -\left(\frac{di_{c'}(c;\pi)}{dc} - \frac{di_i(c)}{dc}\right)\frac{\partial\Phi_p}{\partial i}\frac{\partial\Psi_n}{\partial i} > 0.$$

Therefore, $di/d\pi < 0$ for all $\pi \in (\beta, \infty)$. ■

⊙ **For** $f(i^{**})/i^{**} < 1$**:** The possible cases are either $\mu > 0$ and $\theta > 0$ or $\mu > 0$ and $\theta = 0$, because $\mu = \theta = 0$ is not possible when $f(i^{**})/i^{**} < 1$, as shown by Step 1 in the proof of Proposition 2. Suppose $\theta > 0$ and $\mu > 0$. Then, the equilibrium c, i must satisfy $\Psi_n(c, i; \pi) = 0$ and

$$\Phi_b(c, i; \pi) \quad \equiv \quad \frac{f'(i) - (1-\sigma)k_i(\varrho c, i) - 1}{1 - f'(i)} - \frac{\pi}{\beta} + 1 = 0$$

that is constructed by (12), (13), (24) with $c^s = \varrho c$. Observe

$$\frac{\partial\Phi_b}{\partial c} = -\frac{(1-\sigma)\varrho k_{ic}}{1-f'(i)} > 0, \quad \frac{\partial\Phi_b}{\partial i} = -\frac{(1-\sigma)\left\{k_{ii}(1-f'(i)) + k_i(\cdot)f''(i)\right\}}{(1-f'(i))^2} < 0.$$

for $i > f'^{-1}(1)$. This implicit equation defines an implicit function $i_{i'} = i_{i'}(c;\pi) \in (f'^{-1}(1), \infty)$, for $c \in [0, \infty)$ that satisfies

$$\frac{di_{i'}(c;\pi)}{dc} = -\frac{\varrho k_{ic}(1-f'(i))}{k_{ii}(1-f'(i)) + k_i(\cdot)f''(i)} > 0,$$

$i_{i'}(c^{**}) = \bar{i}_{i'} \in (0, i^{**})$ (as $\pi > \beta$) and $i_{i'}(0) = \underline{i}_{i'} \in (f'^{-1}(1), \bar{i}_{i'}) > 0$. The equilibrium solution can be identified by finding a fixed point of

$$i_{c'}(c;\pi) = i_{i'}(c;\pi)$$

where $i_{c'}(c;\pi) \in [f'^{-1}(1), \infty)$ is defined before and satisfies $\Psi_n(c, i_{c'}(c,\pi);\pi) = 0$. Notice that: (i) $i_{c'}(c_\pi) = f'^{-1}(1) < \underline{i}_{i'} < i_{i'}(c_\pi)$; (ii) $i_{c'}(c^{**}) > i^{**} > \bar{i}_{i'} = i_{i'}(c^{**})$; (ii) $\frac{di_{c'}(c,;\pi)}{dc} - \frac{di_{i'}(c;\pi)}{dc} =$

$$\frac{u''(c)k_c(\cdot)\left\{k_{ii}(1-f'(i)) + k_i f''(i)\right\} - \varrho u'(c)k_i k_{cc} f''(i) - \varrho u'(c)(1-f'(i))(k_{ii}k_{cc} - k_{ic}^2)}{u'(c)k_{ci}\left\{k_{ii}(1-f'(i)) + k_i f''(i)\right\}} > 0.$$

Therefore, given $\mu, \theta > 0$ there exists a unique solution that satisfies $c \in (c_\pi, c^{**})$ and $i \in (f'^{-1}(1), i^{**})$ for all $\pi \in (\beta, \infty)$.

We now prove the latter half of the claims in Proposition 4. Observe that this solution $i = i(\pi)$ satisfies $i(\pi) \to i^{**}$ as $\pi \to \beta$ and $i(\pi) \to f'^{-1}(1)$ as $\pi \to \infty$. As shown in the Step 2 in the proof of Proposition 2, when $f(i^{**})/i^{**} < 1$ this implies that there exists $\tilde{\pi} \in (\beta, \infty)$, such that $\hat{i} = i(\tilde{\pi}) = f(\hat{i}) \in (f'^{-1}(1), i^{**})$ and $f(i) < i$ for $\pi \in (\beta, \tilde{\pi})$ and $f(i) \geq i$ for $\pi \in [\tilde{\pi}, \infty)$. The uniqueness of this critical value $\tilde{\pi}$ can be shown by the monotonicity of $i(\pi)$: for any $\pi \in (\beta, \infty)$,

$$\frac{di(\pi)}{d\pi} = -\frac{\frac{\partial\Psi_n}{\partial c}\frac{\partial\Phi_b}{\partial\pi} - \frac{\partial\Phi_b}{\partial c}\frac{\partial\Psi_n}{\partial\pi}}{\frac{\partial\Phi_b}{\partial i}\frac{\partial\Psi_n}{\partial c} - \frac{\partial\Phi_b}{\partial c}\frac{\partial\Psi_n}{\partial i}} < 0$$

since both the numerator and denominator of the above expression are positive. Therefore, $\mu, \theta > 0$ for $\pi \in (\beta, \tilde{\pi})$ where $i(\pi)$ is decreasing in π. For $\pi \in [\tilde{\pi}, \infty)$, the only possibility is $\mu > 0$ and $\theta = 0$ in which case a unique solution $i = \hat{i}$, which is constant for all $\pi \in [\tilde{\pi}, \infty)$, is pinned down by (6). Finally, $dc(\pi)/d\pi < 0$ for all $\pi \in (\beta, \infty)$ can be shown by the same procedure as before. ■