Computation of sharp bounds on the expected value of a supermodular function of risks with given marginals

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Abstract

We show that the rearrangement algorithm introduced in Puccetti and Rüschendorf (2012a) to compute distributional bounds can be used also to compute sharp lower and upper bounds on the expected value of a supermodular function of d random variables having fixed marginal distributions. Compared to the analytical methods existing in the literature the algorithm is widely applicable, more easily obtained and gives insight into the dependence structures attaining the bounds.

Key words: Moment bounds for dependent risks, distribution functions, rearrangements
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1. Introduction and preliminaries

Let $X_1, \dots, X_d$ be d real-valued random variables on some probability space $(\Omega, \mathfrak{A}, P)$. Given a supermodular function $\psi : \mathbb{R}^d \to \mathbb{R}$, we compute numerically sharp lower and upper bounds on $\mathbb{E}[\psi(X_1, \dots, X_d)]$, where we assume that each X_j has known distribution $F_j(x) = P(X_j \le x)$, $1 \le j \le d$, but the dependence structure of the vector $(X_1, \dots, X_d)'$ is unknown. Thus, we study the problems

$$s_\psi = \inf\left\{\mathbb{E}[\psi(X_1, \dots, X_d)] : X_j \sim F_j, 1 \le j \le d\right\},$$
$$S_\psi = \sup\left\{\mathbb{E}[\psi(X_1, \dots, X_d)] : X_j \sim F_j, 1 \le j \le d\right\}.$$

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While it is well known that the upper bound S_ψ is attained when the d random variables $X_1, \ldots, X_d$ are *comonotonic*, i.e. similarly ordered, the solution of s_ψ is in general open for $d \geq 3$. For $d = 2$ the lower bound s_ψ is attained when the two random variables X_1, X_2 are *countermonotonic*, i.e. oppositely ordered. For $d \geq 2$ the value of s_ψ has been recently given in the paper Wang and Wang (2011) only for the case of identically distributed risks with monotone densities and for a restricted class of supermodular functionals.

Puccetti and Rüschendorf (2012a) introduce the rearrangement algorithm (RA in the following) in order to compute bounds on the distribution function of $\psi(X_1, \ldots, X_d)$. In this paper, we show that the same algorithm can be used to approximate the moment bounds s_ψ and S_ψ for a broad class of supermodular function ψ. Compared to the analytical method described in Wang and Wang (2011), the RA is particularly simple, fast and widely applicable: it can handle inhomogeneous set of marginal distributions and dimensions d in the several hundreds. The RA turns out to be relevant for practical applications also in the computation of S_ψ which often poses serious problems in the case of large vectors of inhomogeneous marginals. Moreover, the RA confirms the results obtained in Wang and Wang (2011) and gives also insight into an analytical solution of s_ψ for arbitrary marginal distributions.

1.1. *Notation*

Let $\boldsymbol{X} = (x_{i,j}) \in \mathbb{R}$ be a $(n \times d)$-matrix. Let $\boldsymbol{X}_{-j}$ be the $(n \times (d-1))$-matrix obtained from $\boldsymbol{X}$ by deleting its j-th column $\boldsymbol{X}_{(j)}$. Denote by $+(\boldsymbol{X})$ and $+(\boldsymbol{X}_{-j})$ the n-dimensional vectors having as components by the componentwise sum of each row of $\boldsymbol{X}$, respectively $\boldsymbol{X}_{-j}$. Formally,

$$+(\boldsymbol{X}) = \begin{pmatrix} x_{1,1} + \cdots + x_{1,d} \\ \vdots \\ x_{i,1} + \cdots + x_{i,d} \\ \vdots \\ x_{n,1} + \cdots + x_{n,d} \end{pmatrix}, +(\boldsymbol{X}_{-j}) = \begin{pmatrix} x_{1,1} + \cdots + x_{1,j-1} + x_{1,j+1} + \cdots + x_{1,d} \\ \vdots \\ x_{i,1} + \cdots + x_{i,j-1} + x_{i,j+1} + \cdots + x_{i,d} \\ \vdots \\ x_{n,1} + \cdots + x_{n,j-1} + x_{n,j+1} + \cdots + x_{n,d} \end{pmatrix}. \tag{1.1}$$

Of course, we have that

$$+(\boldsymbol{X}) = +(\boldsymbol{X}_{-j}) + \boldsymbol{X}_{(j)},\ 1 \leq j \leq d. \tag{1.2}$$

We define $\mathcal{P}(\boldsymbol{X})$ as the set of all $(n \times d)$-matrices obtained from $\boldsymbol{X}$ by rearranging the elements within a number of its columns in a different order, that is

$$\mathcal{P}(\boldsymbol{X}) = \left\{ \tilde{\boldsymbol{X}} = (\tilde{x}_{i,j}) : \tilde{x}_{i,j} = x_{\pi_j(i),j}, \pi_1, \ldots, \pi_d \text{ are permutations of } \{1, \ldots, n\} \right\}.$$

We call each matrix in $\mathcal{P}(\boldsymbol{X})$ a *rearrangement* of $\boldsymbol{X}$.

Given a vector $\boldsymbol{a} \in \mathbb{R}^n$, we denote by $\boldsymbol{a}_{[i]}$ the i-largest component of $\boldsymbol{a}$ ($\boldsymbol{a}_{[n]}$ is the minimal). The vector $\boldsymbol{a}^{\uparrow} = (a_{[1]}, \ldots, a_{[n]})'$ is called the increasing rearrangement of $\boldsymbol{a}$ and the vector $\boldsymbol{a}^{\downarrow} = (a_{[n]}, \ldots, a_{[1]})'$ the decreasing rearrangement of $\boldsymbol{a}$. We write $\boldsymbol{a} \perp \boldsymbol{b}$ to indicate that the components of the vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^n$ are oppositely ordered, that is $(a_j - a_k)(b_j - b_k) \leq 0$ for all $1 \leq j, k \leq n$. For example, we have that $\boldsymbol{a}^{\uparrow} \perp \boldsymbol{a}^{\downarrow}$. *Majorization* between two vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^n$ is defined as

$$\boldsymbol{a} \lesssim \boldsymbol{b} \quad \text{iff} \quad \sum_{i=1}^{j} a_{[i]} \leq \sum_{i=1}^{j} b_{[i]},\ 1 \leq j \leq n, \quad \text{and} \quad \sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i.$$

2. A rearrangement problem

In this section we describe a rearrangement problem which will turn out to be strictly connected to the computation of s_ψ and S_ψ. A similar treatment applied to the solution of a multidimensional assignment problem can be found in Rüschendorf (1983a).

For a function $\psi : \mathbb{R}^d \to \mathbb{R}$ and a $(n \times d)$-matrix $\boldsymbol{X}$, we define the operator $E(\boldsymbol{X})$ as the sum of the n values obtained by applying the function ψ to each row of $\boldsymbol{X}$, i.e.

$$E(\boldsymbol{X}) = \sum_{i=1}^{n} \psi(x_{i,1}, \ldots, x_{i,d}).$$

In this section, we investigate the problem of finding the rearrangements of $\boldsymbol{X}$ which minimize/maximize $E(\boldsymbol{X})$. Formally, we study the problems

$$m_\psi(\boldsymbol{X}) = \min_{\tilde{\boldsymbol{X}} \in \mathcal{P}(\boldsymbol{X})} E(\tilde{\boldsymbol{X}}) \quad \text{and} \quad M_\psi(\boldsymbol{X}) = \max_{\tilde{\boldsymbol{X}} \in \mathcal{P}(\boldsymbol{X})} E(\tilde{\boldsymbol{X}}). \tag{2.1}$$

Throughout the paper we will consider the case that $\psi : \mathbb{R}^d \to \mathbb{R}$ is a supermodular function, i.e.

$$\psi(\boldsymbol{x} \wedge \boldsymbol{y}) + \psi(\boldsymbol{x} \vee \boldsymbol{y}) \geq \psi(\boldsymbol{x}) + \psi(\boldsymbol{y}), \text{ for all } \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d, \tag{2.2}$$

where $\boldsymbol{x} \wedge (\vee)\, \boldsymbol{y}$ is the componentwise minimum (maximum) of $\boldsymbol{x}$ and $\boldsymbol{y}$. The reader is referred to Marshall et al. (2011, Section 6.D) and Block et al. (1989) for equivalent definitions, properties and examples of supermodular functions.

A well-known result due to Lorentz (1953) (see also 6.E.1 in Marshall et al. (2011)) shows that $\boldsymbol{X}^\uparrow$, the *comonotonic rearrangement* of $\boldsymbol{X}$ having all its columns arranged in increasing order, is a solution of $M_\psi(\boldsymbol{X})$ if and only if ψ belongs to $\mathcal{S}_d$, the set of all supermodular functions on $\mathbb{R}^d$.

Proposition 2.1 *For any $(n \times d)$-matrix $\boldsymbol{X}$, we have that*

$$E(\boldsymbol{X}) \leq E(\boldsymbol{X}^\uparrow).$$

if and only if $\psi \in \mathcal{S}_d$.

Proposition 2.1 states that $M_\psi(\boldsymbol{X}) = E(\boldsymbol{X}^\uparrow)$ for any supermodular function ψ. In general it is more difficult to solve $m_\psi(\boldsymbol{X})$.

2.1. *Restriction to convex functions of a sum*

In this subsection, we restrict to considering the particular class of supermodular functions $\mathcal{S}_d^+$ defined as

$$\mathcal{S}_d^+ = \left\{ \psi : \mathbb{R}^d \to \mathbb{R} : \psi(x_1, \ldots, x_d) = f(x_1 + \cdots + x_d), \text{ for some convex } f \right\}.$$

For $\psi \in \mathcal{S}_d^+$, we have

$$E(\boldsymbol{X}) = \sum_{i=1}^{n} f(x_{i,1} + \cdots + x_{i,d}).$$

Now define

$$O_+(\boldsymbol{X}) = \left\{ \tilde{\boldsymbol{X}} \in \mathcal{P}(\boldsymbol{X}) : \tilde{\boldsymbol{X}}_{(j)} \perp +(\tilde{\boldsymbol{X}}_{-j}),\ 1 \leq j \leq d \right\}$$

be the set of those rearrangement $\tilde{\boldsymbol{X}}$ of $\boldsymbol{X}$ having each column oppositely ordered to the sum of the others. Based on two well-known results on rearrangements, it is possible to restrict the domain of the min problem in (2.1) to the smaller set $O_+(\boldsymbol{X})$.

Proposition 2.2 (Day (1972)) *For any two vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, we have that $\boldsymbol{x}^{\uparrow} + \boldsymbol{y}^{\downarrow} \lesssim \boldsymbol{x} + \boldsymbol{y}$.*
Proposition 2.3 (Hardy et al. (1929)) *For any convex function f, $\tilde{\boldsymbol{y}} \lesssim \tilde{\boldsymbol{x}}$ implies that*

$$\sum_{i=1}^{n} f(\tilde{y}_i) \leq \sum_{i=1}^{n} f(\tilde{x}_i). \tag{2.3}$$

Proposition 2.4 *If $\psi \in \mathcal{S}_d^+$, we have that*

$$m_\psi(\boldsymbol{X}) = \min_{\tilde{\boldsymbol{X}} \in O_+(\boldsymbol{X})} E(\tilde{\boldsymbol{X}}).$$

Proof. For any $\tilde{\boldsymbol{X}} \notin O_+(\boldsymbol{X})$, it is possible to find an index j so that $\tilde{\boldsymbol{X}}_{(j)}$ is not oppositely ordered to $+(\tilde{\boldsymbol{X}}_{-j})$. Denote by $\tilde{\boldsymbol{Y}} \in \mathcal{P}(\boldsymbol{X})$ the matrix obtained from $\tilde{\boldsymbol{X}}$ by rearranging its j-th column oppositely to the sum of the others. Using Proposition 2.2 and (1.2) we have that

$$+(\tilde{\boldsymbol{Y}}) = +(\tilde{\boldsymbol{X}}_{-j})^{\uparrow} + \tilde{\boldsymbol{X}}_{(j)}^{\downarrow} \lesssim +(\tilde{\boldsymbol{X}}_{-j}) + \tilde{\boldsymbol{X}}_{(j)} = +(\tilde{\boldsymbol{X}}) = \tilde{\boldsymbol{x}}.$$

Let $\tilde{\boldsymbol{y}} = +(\tilde{\boldsymbol{Y}})$. By Proposition 2.3, $\tilde{\boldsymbol{y}} \lesssim \tilde{\boldsymbol{x}}$ implies that

$$\sum_{i=1}^{n} f(\tilde{y}_i) \leq \sum_{i=1}^{n} f(\tilde{x}_i),$$

for any convex function f. It follows that

$$E(\tilde{\boldsymbol{Y}}) = \sum_{i=1}^{n} f(\tilde{y}_{i,1} + \cdots + \tilde{y}_{i,d}) = \sum_{i=1}^{n} f(\tilde{y}_i) \leq \sum_{i=1}^{n} f(\tilde{x}_i) = E(\tilde{\boldsymbol{X}}). \tag{2.4}$$

As noted at the end of the proof of Theorem 2.1 in Puccetti and Rüschendorf (2012a), being the set $\mathcal{P}(\boldsymbol{X})$ finite, it is possible to pass from any $\tilde{\boldsymbol{X}} \notin O_+(\boldsymbol{X})$ to a matrix $\boldsymbol{X}^* \in O_+(\boldsymbol{X})$ in a finite number of steps. Considering (2.4), we can restrict the domain of the min problem $m_\psi(\boldsymbol{X})$ to the set $O_+(\boldsymbol{X})$. □

At this point, the rearrangement algorithm introduced in Puccetti and Rüschendorf (2012a) can be used to find elements in $O_+(\boldsymbol{X})$ which are, by Proposition 2.4, candidate solutions to $m_\psi(\boldsymbol{X})$.

Rearrangement algorithm to find elements in $O_+(\boldsymbol{X})$. Start with any $\tilde{\boldsymbol{X}} \in \mathcal{P}(\boldsymbol{X})$. Define $\tilde{\boldsymbol{X}}_1$ by iteratively rearranging its j-th column $\tilde{\boldsymbol{X}}_{(j)}$ such that $\tilde{\boldsymbol{X}}_{(j)} \perp +(\tilde{\boldsymbol{X}}_{-j})$, for $1 \leq j \leq d$. Then, repeat using $\tilde{\boldsymbol{X}}_1$ as the initial matrix until an element $\boldsymbol{X}^* \in O_+(X)$ is found.

We note that not all the matrices in $O_+(\boldsymbol{X})$ are optimal. As an example one can consider the matrices

$$\boldsymbol{X} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \\ 4 & 4 & 4 \\ 5 & 5 & 5 \end{bmatrix}, \quad \tilde{\boldsymbol{X}} = \begin{bmatrix} 5 & 1 & 2 \\ 3 & 5 & 1 \\ 2 & 3 & 4 \\ 4 & 2 & 3 \\ 1 & 4 & 5 \end{bmatrix} \quad \text{and} \quad \tilde{\boldsymbol{Y}} = \begin{bmatrix} 1 & 5 & 3 \\ 2 & 3 & 4 \\ 3 & 1 & 5 \\ 4 & 4 & 1 \\ 5 & 2 & 2 \end{bmatrix}.$$

Even if $\tilde{\boldsymbol{X}}, \tilde{\boldsymbol{Y}} \in O_+(\boldsymbol{X})$, the vector $\tilde{\boldsymbol{x}} = +(\tilde{\boldsymbol{X}})$ is strictly larger than the vector $\tilde{\boldsymbol{y}} = +(\tilde{\boldsymbol{Y}})$ with respect to $\lesssim$. It follows that for instance for the stop-loss functional $\psi(x_1, \ldots, x_d) = [x_1 + \cdots + x_d - 9]^+$

we have that $E(\tilde{\boldsymbol{Y}}) = 0 < 1 = E(\tilde{\boldsymbol{X}})$. In this case, the matrix $\tilde{\boldsymbol{X}} \in O_+(\boldsymbol{X})$ does not attain $m_\psi(\boldsymbol{X})$. In applications to follow we will however see that in many cases any element in $O_+(\boldsymbol{X})$ gives a good approximation to the optimal solution.

2.2. *Extensions to general supearadditive function ψ*

A natural question is whether the method described in Section 2.1 can be extended to find solutions/approximations of $m_\psi(\boldsymbol{X})$ for a general supermodular function ψ. The answer is positive, provided that ψ satisfies the following extra requirement. We assume that there exist two measurable supermodular functions $\psi^{d-1} : \mathbb{R}^{d-1} \to \mathbb{R}$ and $\psi^2 : \mathbb{R}^2 \to \mathbb{R}$ such that ψ satisfies

$$\psi(x_1, \ldots, x_d) = \psi^2(x_j, \psi^{d-1}(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)), \ 1 \le j \le d. \tag{2.5}$$

Relevant cases of supermodular functions ψ satisfying (2.5) are the sum ($\psi^2(x_1, x_2) = x_1 + x_2$), the product ($\psi^2(x_1, x_2) = x_1 x_2$, for $x_1, x_2 > 0$), the min ($\psi^2(x_1, x_2) = \min\{x_1, x_2\}$) and the $-\max$ ($\psi^2(x_1, x_2) = -\max\{x_1, x_2\}$) operators. Asymmetric functions do not satisfy (2.5).

We now extend to a general function ψ the definitions given in (1.1) in the case of the sum operator. Denote by $\boldsymbol{\Psi}(\boldsymbol{X})$ (respectively, $\boldsymbol{\Psi}(\boldsymbol{X}_{-j})$) the n-dimensional vectors obtained by applying the function ψ (resp., ψ^{d-1}), to each row of $\boldsymbol{X}$ (resp., $\boldsymbol{X}_{-j}$). Analogously to (1.1), we have

$$\boldsymbol{\Psi}(\boldsymbol{X}) = \begin{pmatrix} \psi(x_{1,1}, \ldots, x_{1,d}) \\ \vdots \\ \psi(x_{i,1}, \ldots, x_{i,d}) \\ \vdots \\ \psi(x_{n,1}, \ldots, x_{n,d}) \end{pmatrix}, \boldsymbol{\Psi}(\boldsymbol{X}_{-j}) = \begin{pmatrix} \psi^{d-1}(x_{1,1}, \ldots, x_{1,j-1}, x_{1,j+1}, \ldots, x_{1,d}) \\ \vdots \\ \psi^{d-1}(x_{i,1}, \ldots, x_{i,j-1}, x_{i,j+1}, \ldots, x_{i,d}) \\ \vdots \\ \psi^{d-1}(x_{n,1}, \ldots, x_{n,j-1}, x_{n,j+1}, \ldots, x_{n,d}) \end{pmatrix}.$$

Now let

$$O_\psi(\boldsymbol{X}) = \left\{ \tilde{\boldsymbol{X}} \in \mathcal{P}(X) : \tilde{\boldsymbol{X}}_{(j)} \perp \boldsymbol{\Psi}(\tilde{\boldsymbol{X}}_{-j}), 1 \le j \le d \right\},$$

be the set of those permutation matrices $\tilde{\boldsymbol{X}}$ such that $\tilde{\boldsymbol{X}}_{(j)}$ is oppositely ordered to $\boldsymbol{\Psi}_{-j}(\tilde{\boldsymbol{X}})$ for all $1 \le j \le d$. Similarly to Proposition 2.4 we can restrict the domain of the min problem $m_\psi(\boldsymbol{X})$ to the set $O_\psi(\boldsymbol{X})$. We use the following result due to Lorentz (1953); see also 6.E.1 in Marshall et al. (2011).

Proposition 2.5 (Lorentz (1953)) *For any vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^n$, we have*

$$\sum_{i=1}^{n} \phi(a_{[i]}, b_{[n-i+1]}) \le \sum_{i=1}^{n} \phi(a_i, b_i)$$

if and only if $\phi \in \mathcal{S}_d$.

Proposition 2.6 *If $\psi \in \mathcal{S}_d$ is coordinatewise strictly monotonic and satisfies condition* (2.5)*, we have that*

$$m_\psi(\boldsymbol{X}) = \min_{\tilde{\boldsymbol{X}} \in O_\psi(\boldsymbol{X})} E(\tilde{\boldsymbol{X}}).$$

Proof. For any $\tilde{\boldsymbol{X}} \notin O_\psi(\boldsymbol{X})$, it is possible to find an index j so that $\tilde{\boldsymbol{X}}_{(j)}$ is not oppositely ordered to $\boldsymbol{\Psi}(\tilde{\boldsymbol{X}}_{-j})$. Let $\tilde{\boldsymbol{Y}} \in \mathcal{P}(\boldsymbol{X})$ be the matrix obtained from $\tilde{\boldsymbol{X}}$ by rearranging its j-th column oppositely

to $\boldsymbol{\Psi}(\tilde{\boldsymbol{X}}_{-j})$. Applying Proposition 2.5 to the vectors $\boldsymbol{a} := \boldsymbol{\Psi}(\tilde{\boldsymbol{X}}_{-j}), \boldsymbol{b} := \tilde{\boldsymbol{X}}_{(j)}$ and using (2.5) we obtain that

$$\begin{aligned} E(\tilde{\boldsymbol{Y}}) &= \sum_{i=1}^{n} \psi(\tilde{y}_{i,1}, \ldots, \tilde{y}_{i,n}) = \sum_{i=1}^{n} \psi^2(a_{[i]}, b_{[n-j+1]}) \leq \sum_{i=1}^{n} \psi^2(a_i, b_i) \\ &= \sum_{i=1}^{n} \psi^2(\tilde{x}_{i,j}, \psi^{d-1}(\tilde{x}_{i,1}, \ldots, \tilde{x}_{i,j-1}, \tilde{x}_{i,j+1}, \ldots, \tilde{x}_{i,n})) = \sum_{i=1}^{n} \psi(\tilde{x}_{i,1}, \ldots, \tilde{x}_{i,n}) = E(\tilde{\boldsymbol{X}}). \quad (2.6) \end{aligned}$$

As noted at the end of the proof of Theorem 2.1 in Puccetti and Rüschendorf (2012a), being the set $\mathcal{P}(\boldsymbol{X})$ finite and ψ strictly monotonic, it is possible to pass from any $\tilde{\boldsymbol{X}} \notin O_\psi(\boldsymbol{X})$ to a matrix $\boldsymbol{X}^* \in O_\psi(\boldsymbol{X})$ in a finite number of steps. Considering (2.6), we can restrict the domain of the min problem $m_\psi(\boldsymbol{X})$ to the set $O_\psi(\boldsymbol{X})$. □

The proof of Proposition 2.6 indicates that the rearrangement algorithm can be used with any supermodular function satisfying the extra condition (2.5).

Rearrangement algorithm to find elements in $O_\psi(\boldsymbol{X})$. Start with any $\tilde{\boldsymbol{X}} \in \mathcal{P}(\boldsymbol{X})$. Define $\tilde{\boldsymbol{X}}_1$ by iteratively rearranging its j-th column $\tilde{\boldsymbol{X}}_{(j)}$ such that $\tilde{\boldsymbol{X}}_{(j)} \perp \boldsymbol{\Psi}(\tilde{\boldsymbol{X}}_{-j})$, for $1 \leq j \leq d$. Then, repeat using $\tilde{\boldsymbol{X}}_1$ as the initial matrix until an element $\boldsymbol{X}^* \in O_\psi(\boldsymbol{X})$ is found.

Remark 2.7 *We conclude this section by summarizing some important points.*

- *(i) Proposition 2.6 is not an extension of Proposition 2.4. Indeed, there exist supermodular functions $\psi \in \mathcal{S}_d^+$ which do not satisfy condition* (2.5)*. An example is given by the stop-loss functional $\psi(x_1, \ldots, x_d) = [x_1 + \cdots + x_d - k]^+$, for some $k \neq 0$. This also explains why we need majorization to obtain Proposition 2.4.*
- *(ii) A rearrangement matrix $\boldsymbol{X}^*$ is a solution of $m_\psi(\boldsymbol{X}) = E(\boldsymbol{X}^*)$ for some function $\psi \in \mathcal{S}_d^+$ if and only if $\boldsymbol{X}^*$ is a solution of $m_\psi(\boldsymbol{X}) = E(\boldsymbol{X}^*)$ for all functions $\psi \in \mathcal{S}_d^+$. Analogously, $M_\psi(\boldsymbol{X}) = E(\boldsymbol{X}^\uparrow)$ for any $\psi \in \mathcal{S}_d$.*
- *(iii) In the general case that $\psi \in \mathcal{S}_d$ a solution of $m_\psi(\boldsymbol{X})$ may depend on the function ψ.*
- *(iv) Proposition 2.6 extends to all $\psi \in \mathcal{S}_d$ only when $d = 2$. In this case, condition* (2.5) *is automatically satisfied. For $d = 2$, denote by $\boldsymbol{X}^\downarrow$ a* countermonotonic rearrangement *of $\boldsymbol{X}$ having the two columns arranged in opposite order. Proposition 2.5 implies that for any $(n \times 2)$-matrix $\boldsymbol{X}$ we have that*

$$E(\boldsymbol{X}) \geq E(\boldsymbol{X}^\downarrow).$$

  if and only if $\psi \in \mathcal{S}_d$. However, the case $d = 2$ is seldom relevant in applications.
- *(v) The rearrangement algorithm can be used also when the function $\psi \in \mathcal{S}_d$ is non-strictly monotonic, provided that the set $O_\psi(\boldsymbol{X})$ is nonempty.*

3. Moment bounds

Given a set of marginal distributions $F_1, \dots, F_d$ and a supermodular function $\psi : \mathbb{R}^d \to \mathbb{R}$, the aim of this paper is to compute

$$s_\psi = \inf \left\{ \mathbb{E}[\psi(X_1, \dots, X_d)] : X_j \sim F_j, 1 \le j \le d \right\}, \tag{3.1a}$$

$$S_\psi = \sup \left\{ \mathbb{E}[\psi(X_1, \dots, X_d)] : X_j \sim F_j, 1 \le j \le d \right\}. \tag{3.1b}$$

If each marginal distribution F_j is *n-discrete*, that is uniformly distributed on a set of n real values $x_{i,j}, i = 1, \dots, n$, using a rearrangement argument similar to the one given in Puccetti and Rüschendorf (2012a, Section 3) we obtain that

$$s_\psi \simeq m_\psi(\boldsymbol{X})/n \quad \text{and} \quad S_\psi \simeq M_\psi(\boldsymbol{X})/n, \tag{3.2}$$

where $\boldsymbol{X} = (x_{i,j})$. The approximations in (3.2) hold for n large enough only when each F_j is n-discrete but can be used to compute numerically s_ψ and S_ψ also in the general case of arbitrary marginals. Indeed, it is always possible to find two n-discrete distributions which approximate any F_j from below and from above. For instance, we define the discrete distributions $\underline{F}_j$ and $\overline{F}_j$ as

$$\underline{F}_j(x) = \frac{1}{n} \sum_{r=0}^{n-1} 1_{[q_r, +\infty)}(x) \quad \text{and} \quad \overline{F}_j(x) = \frac{1}{n} \sum_{r=1}^{n} 1_{[q_r, +\infty)}(x),$$

where the jump points q_r are defined by $q_r := F_j^{-1}(r/n), 0 \le r \le n$. Since $\underline{F}_j \ge F_j \ge \overline{F}_j$, if we assume that ψ is componentwise increasing we obtain that

$$\underline{s}_\psi \le s_\psi \le \overline{s}_\psi \quad \text{and} \quad \underline{S}_\psi \le S_\psi \le \overline{S}_\psi. \tag{3.3}$$

where $\underline{s}_\psi$ (respectively $\overline{s}_\psi$) is the analogous of (3.1a) when $F_j = \underline{F}_j$ (resp. $F_j = \overline{F}_j$). Analogously, $\underline{S}_\psi$ (resp. $\overline{S}_\psi$) is the analogous of (3.1b) when $F_j = \underline{F}_j$(resp. $F_j = \overline{F}_j$).

We denote by $\underline{\boldsymbol{X}} = (\underline{x}_{i,j})$ (resp. $\overline{\boldsymbol{X}} = (\overline{x}_{i,j})$) the $(n \times d)$-matrix having as j-th column the vector of jump points of the distribution $\underline{F}_j$ (resp. $\overline{F}_j$) , i.e.

$$\underline{x}_{i,j} = F_j^{-1}\left(\frac{i-1}{n}\right) \quad \text{and} \quad \overline{x}_{i,j} = F_j^{-1}\left(\frac{i}{n}\right), \ 1 \le i \le n.$$

Using (3.2) and (3.3) we obtain, for n large enough, that

$$m_\psi(\underline{\boldsymbol{X}})/n \le s_\psi \le m_\psi(\overline{\boldsymbol{X}})/n, \tag{3.4a}$$

$$M_\psi(\underline{\boldsymbol{X}})/n \le S_\psi \le M_\psi(\overline{\boldsymbol{X}})/n. \tag{3.4b}$$

Recalling Proposition 2.1, the following numerical range on S_ψ directly follow from (3.4b):

$$E(\underline{\boldsymbol{X}}^{\uparrow})/n \le S_\psi \le E(\overline{\boldsymbol{X}}^{\uparrow})/n. \tag{3.5}$$

The rearrangement algorithms described in Section 2 can be used in combination with (3.4a) to find a numerical range also for s_ψ. Using Proposition 2.6 or Proposition 2.4 (depending on the properties of ψ), we obtain that

$$E(\tilde{\boldsymbol{Y}})/n \simeq s_\psi \le E(\tilde{\boldsymbol{Z}})/n, \tag{3.6}$$

for any $\tilde{\boldsymbol{Y}} \in O_\psi(\underline{\boldsymbol{X}})$ and $\tilde{\boldsymbol{Z}} \in O_\psi(\overline{\boldsymbol{X}})$. Note that the right-hand inequality in (3.6) is always satisfied but the left-hand one may fail if the matrix $\tilde{\boldsymbol{Y}}$ does not attain $m_\psi(\underline{\boldsymbol{X}})$. However, in practice the

numerical range in (3.6) always turns out to yield a good approximation of s_ψ. At this point two matrices $\tilde{\boldsymbol{Y}} \in \mathcal{O}_\psi(\underline{\boldsymbol{X}})$ and $\tilde{\boldsymbol{Z}} \in \mathcal{O}_\psi(\overline{\boldsymbol{X}})$ producing a range as in (3.6) can be found using the rearrangement algorithms described in Section 2.

It is important to note that if a distribution F_j is unbounded from above, that is $F_j^{-1} = +\infty$, and also ψ is unbounded from above, we obtain that $M_\psi(\overline{\boldsymbol{X}}) = m_\psi(\overline{\boldsymbol{X}}) = +\infty$ so the upper limits of the ranges in (3.5) and (3.6) are not useful. However, we will see in practice that the corresponding lower limits turn out to be sufficiently accurate for high values of n. Similar considerations hold if F_j and ψ are unbounded from below.

If $\mathbb{E}[\psi(X_1, \dots, X_d)]$ is finite, the accuracy of the numerical ranges given in (3.5) and in (3.6) can be increased by choosing:

- a larger value of n, so that the approximation to F_j given by the discrete distributions $\underline{F}_j$ and $\overline{F}_j$ is more accurate and the transition from continuous to discrete rearrangements in (3.2) is justified; see Puccetti and Rüschendorf (2012a, Section 3).
- a number of different random starting rearrangement matrices for the RA in order to find different elements in the set $\mathcal{O}_\psi(\boldsymbol{X})$.

Having mainly applications to quantitative risk management in mind, in the following we will always compute the ranges (3.5) and (3.6) for continuous marginal distributions F_j. In these cases, we always find that the ranges in (3.5) and (3.6) yield a very good approximation of S_ψ and, respectively s_ψ, with a single starting matrix and a high value for n.

4. Applications

In this section, we compute the numerical ranges in (3.5) and in (3.6) for different increasing, supermodular functionals ψ and different sets of marginals $F_j, 1 \le j \le d$. In Table 1, we compute (3.5) and (3.6) for the product ($\psi = \times$) of d random variables being all uniformly distributed on $[0, 1]$. In this case, it is easy to see that $S_\times = 1/(d+1)$ while the lower bound $s_\times$ has been given analytically in the recent paper Wang and Wang (2011). In Table 1 we report also the computation times of the range (3.6) (the computation of (3.5) is immediate for any dimension d) obtained on an Apple MacBook Air (2 GHz Intel Core i7, 8 GB RAM) by setting $n = 10^5$.

d	avg time (secs)	$s_\times$ (RA range)	$s_\times$ (analytical)	$S_\times$ (RA range)	$S_\times$ (analytical)
3	7	$5.4800 \times 10^{-2} - 5.4807 \times 10^{-2}$	5.4803×10^{-2}	0.2500-0.2500	0.2500
4	8	$1.9096 \times 10^{-2} - 1.9100 \times 10^{-2}$	1.9098×10^{-2}	0.2000-0.2000	0.2000
5	9	$6.8594 \times 10^{-3} - 6.8615 \times 10^{-3}$	6.8604×10^{-3}	0.1667-0.1667	0.1667
10	18	$4.5385 \times 10^{-5} - 4.5435 \times 10^{-5}$	4.5410×10^{-5}	0.0909-0.0909	0.0909
20	46	$2.0553 \times 10^{-9} - 2.0639 \times 10^{-9}$	2.0612×10^{-9}	0.0476-0.0476	0.0476
50	188	$1.8865 \times 10^{-22} - 1.9352 \times 10^{-22}$	1.9287×10^{-22}	0.0196-0.0196	0.0196
100	595	$3.3851 \times 10^{-44} - 3.745 \times 10^{-44}$	3.7201×10^{-44}	0.0099-0.0099	0.0099

Table 1
Numerical ranges (see (3.5) and (3.6)) and analytical values for $s_\times$ and $S_\times$ for the product of d random variables uniformly distributed on $[0, 1]$. Numerical ranges are computed via the RA with $n = 10^5$, while analytical values are taken from Table 4.1 in Wang and Wang (2011). Computation times of (3.6) are also reported.

The aim of this first example was only to show the accuracy of our method, as for the homogeneous case illustrated in Table 1 the values of $S_\times$ and $s_\times$ can be computed analytically. In the more general case that the marginal distributions are not homogeneous, the situation is different. The analytical results in Wang and Wang (2011) only hold for identically distributed random variables and so far there does not exist a method which allows to compute the lower bound s_ψ analytically for any $\psi \in \mathcal{S}$ in the inhomogeneous case. Apart from some particular cases illustrated in Dhaene et al. (2002a), also the computation of S_ψ may be numerically cumbersome when dealing with inhomogeneous marginals. Being entirely numerical, the algorithm described in this paper can be used with a large number of marginal distributions and for a broad class of supermodular functionals. In order to illustrate the full potential of the RA, in Table 2 we compute sharp lower and upper bounds for the expectation of the product of d inhomogeneous uniformly distributed random variables.

d	avg time (secs)	$s_\times$ (RA range)	$S_\times$ (RA range)
10	18	$1.5470 \times 10^{-1} - 1.5473 \times 10^{-1}$	$4.2191 \times 10^{0} - 4.2194 \times 10^{0}$
20	46	$5.0315 \times 10^{-2} - 5.0333 \times 10^{-2}$	$1.0764 \times 10^{2} - 1.0766 \times 10^{2}$
50	188	$1.6794 \times 10^{-3} - 1.6794 \times 10^{-3}$	$4.8464 \times 10^{6} - 4.8482 \times 10^{6}$
100	595	$5.7255 \times 10^{-6} - 5.7362 \times 10^{-6}$	$6.0091 \times 10^{14} - 6.0133 \times 10^{14}$

Table 2
RA numerical ranges (3.5) and (3.6) for the product of d random variables uniformly distributed on $[a_j, a_j + 1]$, where $a_j = (j-1)/d, 1 \le j \le d$. Numerical ranges have been obtained by setting $n = 10^5$. Computation times of (3.6) are also reported.

As a second application, we compute sharp bounds on the expectation of the stop-loss function $\psi(x_1, \ldots, x_d) = [x_1 + \cdots + x_d - k]^+$ for a number d of Exp(1)-distributed random variables. In this case it is easy to see that $S_\psi = \int_{k/d}^{+\infty} (dx - k)e^{-x}dx$ while the lower bound $s_\times$ can be computed analytically using Theorem 3.5 in Wang and Wang (2011). The results obtained for $n = 10^5$ are collected in Table 3. In this example, being the marginal distributions and the function under study unbounded from above, we compute only the lower approximations of the sharp bounds. Table 3 however shows that these lower approximations are sufficiently accurate. In Table 4 we treat the case of the stop-loss function of $d = 3$ inhomogeneous Pareto-distributed risks. In this latter case analytical values for the sharp bounds are available only for $k = 0$.

k	s_ψ (numerical lb)	s_ψ (analytical)	S_ψ (numerical lb)	S_ψ (analytical)
0	2.9998	3.0000	2.9998	3.000
1	1.9998	2.0000	2.1494	2.1496
2	0.9998	1.0000	1.5401	1.5403
3	0.16939	0.16956	1.1035	1.1036
4	0.057013	0.057159	0.79061	0.79079
5	0.020369	0.020492	0.56645	0.56663

Table 3
Numerical lower bounds (lbs) on s_ψ and S_ψ for the stop-loss function with deductible k for $d = 3$ random variables being all Exp(1)-distributed. Numerical ranges are computed via the RA within 7 seconds with $n = 10^5$. Analytical values for s_ψ and S_ψ, computed via Theorem 3.5 in Wang and Wang (2011), are also given.

k	s_ψ (numerical lower bound)	S_ψ (numerical lower bound)
0	1.828134 (exact=1.833333)	1.828134 (exact=1.833333)
1	0.8281339	1.149902
2	0.4027471	0.8114906
3	0.2829846	0.6144147
4	0.2181444	0.4877124
5	0.1772537	0.4004565

Table 4
Numerical lower bounds (lbs) on s_ψ and S_ψ for the stop-loss function with deductible k for $d = 3$ random variables with distribution $X_j \sim \text{Pareto}(j+1)$. Numerical ranges are computed via the RA within 7 seconds with $n = 10^5$.

The computation time of the RA is not affected by the type of marginal distributions chosen but only depends on their number d and on the accuracy parameter n. The figures obtained in the tables above for $n = 10^5$ can be already considered reasonably accurate. However, an important feature of the algorithm is that it can handle larger values of n and d without heavy memory issues. If extra-accuracy is required, with $n = 10^6$ one can obtain an estimate of $s_\times$ in about 3 minutes for the product of $d = 3$ random variables. If one needs instead less precision, using $n = 10^4$ provides an estimate of $s_\times$ within 20 minutes for the product of $d = 500$ (possibly inhomogeneous) random variables.

5. Rearrangement structures and dependence

For a given $(n \times d)$-matrix $\boldsymbol{X} = (x_{i,j})$, any rearrangement $\tilde{\boldsymbol{X}} \in \mathcal{P}(\boldsymbol{X})$ can be seen as the support of a n-discrete, d-variate distribution giving probability mass $1/n$ to each one of its n row vectors. Under this view, any such $\tilde{\boldsymbol{X}}$ has the same marginal distributions F_j, where F_j is uniformly distributed over the n real values $x_{i,j}$, $1 \le i \le n$. Therefore, any rearrangement matrix represents a different dependence structure coupling the fixed discrete marginal distributions. In particular, $\tilde{\boldsymbol{X}}$ has a copula belonging to the class of shuffles of Min copulas as introduced in Mikusiński et al. (1992) and therefore represents a mutually complete dependence between the fixed marginals in the sense defined in Lancaster (1963). It has been observed that the class of shuffles of Min is dense in the class of copulas endowed with the L^∞-norm. In fact, any copula can be considered as a generalization to the infinite-dimensional space of such rearrangement matrices (see for instance Kolesárová et al. (2006)). Equivalently, any dependence structure can be approximated by a the copula of a rearrangement matrix for n large enough. For more details on the link between the idea of a rearrangement and copulas as dependence structures, we refer to Rüschendorf (1983b) and the more recent paper Durante and Sánchez (2012) which contains an accurate list of references.

On the above grounds, it is of interest to investigate the rearrangement matrices yielding the ranges (3.5) and (3.6). The comonotonic matrix $\boldsymbol{X}^\uparrow$ yielding the range (3.5) represents comonotonic dependence between its columns. Since comonotonic dependence has been well studied in the literature (see for instance Dhaene et al. (2002b) and Rüschendorf (2005)), here it is more interesting to study the structure of the rearrangement matrices yielding the numerical range (3.6).

In the part (A) of Figure 5 we give the matrix $\tilde{\boldsymbol{X}} \in O_+(\underline{\boldsymbol{X}})$ approximating for $n = 50$ the minimal expectation of $d = 3$ homogeneous Exp(1)-marginals. The copula of $\tilde{\boldsymbol{X}}$ approximates

the optimal copula Q_n^P defined in Wang and Wang (2011). The copula Q_n^P describes a structural dichotomy under which either the marginals are *d-completely mixable* (see Wang and Wang (2011) for a definition of complete mixability) or one of the components is large and the others small. One can check that basically the same dependence structure occurs the rows of $\tilde{\boldsymbol{X}}$: either all the components of the row are close to each other, and sum up to a value which is around the threshold 2.7 (e.g. row 14), or one of them is large and the other two are small (e.g. row 23). As noted in point (ii) of Remark 2.7, the same rearrangement structure characterizes any solution of $m_\psi(\boldsymbol{X})$ when $\psi \in \mathcal{S}_d^+$. Moreover, Puccetti and Rüschendorf (2012b) show that the copula Q_n^P maximize the tail function for the sum of d homogeneous random variables with given marginal distributions. In the part (B) of Figure 5 we give the matrix $\tilde{\boldsymbol{Y}} \in O_+(\underline{\boldsymbol{X}})$ approximating for $n = 50$ the miminal expectation of the stop-loss function for the inhomogeneous Pareto portfolio underlying Table 4. The matrix $\tilde{\boldsymbol{Y}}$ shows a structure of dependence similar to the matrix $\tilde{\boldsymbol{X}}$ given in the part (A) of the same figure. The structure of the matrix $\tilde{\boldsymbol{Y}}$ suggests that the optimal coupling results in Wang and Wang (2011) and Puccetti and Rüschendorf (2012b) as well as the concept of *complete mixability* could be extended to the inhomogeneous setting. Finally, In the part (C) of Figure 5 we give the matrix $\tilde{\boldsymbol{Z}} \in O_\psi(\underline{\boldsymbol{X}})$ approximating for $n = 50$ the miminal expectation of the product ($\psi = \times$) of the three uniform marginals underlying Table 1. The matrix $\tilde{\boldsymbol{Z}}$ suggest that the concept of complete mixability could be extended to a broader class of functionals ψ as well.

6. Conclusions and forthcoming research

In this paper, we show that the rearrangement algorithm (RA) introduced in Puccetti and Rüschendorf (2012a) can be used to calculate sharp lower and upper bounds on the expected value of a supermodular function of dependent random variables having fixed marginals. The RA is accurate, fast and can be used to handle random variables with inhomogeneous marginals, in high dimensions. Moreover, the algorithm also gives insight into the dependence structures attaining the bounds. We believe that the numerical moment bounds studied in this paper will have a wide range of application in quantitative risk management. Matthias Scherer (private communication) suggested that they should be relevant in the computation of bounds on the price of multi-assets options. We will investigate this application further in future research.

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(A)	1	2	3	Σ
1	0.73396918	0.44628710	1.51412773	2.694384
2	0.65392647	0.86750057	1.13943428	2.660861
3	0.15082289	2.52572864	0.12783337	2.804385
4	3.91202301	0.00000000	0.02020271	3.932226
5	0.02020271	3.91202301	0.00000000	3.932226
6	0.38566248	1.83258146	0.47803580	2.696280
7	0.30110509	0.54472718	1.83258146	2.678414
8	2.81341072	0.08338161	0.10536052	3.002153
9	1.20397280	0.77652879	0.69314718	2.673649
10	2.12026354	0.24846136	0.32850407	2.697229
11	2.52572864	0.12783337	0.15082289	2.804385
12	0.10536052	2.81341072	0.08338161	3.002153
13	0.41551544	0.30110509	1.96611286	2.682733
14	1.51412773	0.82098055	0.35667494	2.691783
15	0.86750057	0.73396918	1.07880966	2.680279
16	0.57981850	1.71479843	0.41551544	2.710132
17	0.35667494	0.22314355	2.12026354	2.700082
18	1.02165125	1.42711636	0.24846136	2.697229
19	1.07880966	0.69314718	0.91629073	2.688248
20	1.42711636	0.51082562	0.73396918	2.671911
21	3.21887582	0.04082199	0.06187540	3.321573
22	2.30258509	0.19845094	0.22314355	2.724180
23	0.06187540	3.21887582	0.04082199	3.321573
24	0.22314355	0.17435339	2.30258509	2.700082
25	0.96758403	0.38566248	1.34707365	2.700320
26	0.00000000	0.02020271	3.91202301	3.932226
27	0.47803580	0.47803580	1.71479843	2.670870
28	0.27443685	1.20397280	1.20397280	2.682382
29	1.71479843	0.35667494	0.61618614	2.687660
30	0.24846136	1.60943791	0.82098055	2.678880
31	1.60943791	0.27443685	0.77652879	2.660404
32	0.12783337	0.15082289	2.52572864	2.804385
33	1.34707365	0.65392647	0.65392647	2.654927
34	0.32850407	1.34707365	1.02165125	2.697229
35	0.61618614	0.61618614	1.42711636	2.659489
36	0.19845094	2.30258509	0.19845094	2.699487
37	1.27296568	0.91629073	0.51082562	2.700082
38	1.13943428	1.13943428	0.44628710	2.725156
39	0.17435339	2.12026354	0.38566248	2.680279
40	1.83258146	0.32850407	0.54472718	2.705813
41	0.44628710	0.96758403	1.27296568	2.686837
42	0.77652879	1.07880966	0.86750057	2.722839
43	0.54472718	1.96611286	0.17435339	2.685193
44	1.96611286	0.41551544	0.30110509	2.682733
45	0.91629073	1.51412773	0.27443685	2.704855
46	0.69314718	1.02165125	0.96758403	2.682382
47	0.82098055	1.27296568	0.57981850	2.673765
48	0.08338161	0.10536052	2.81341072	3.002153
49	0.04082199	0.06187540	3.21887582	3.321573
50	0.51082562	0.57981850	1.60943791	2.700082
Σ	47.12338332	47.12338332	47.12338332	

(B)	1	2	3	Σ
1	0.06600358	1.154434690	0.093265114	1.313703
2	0.19522861	0.842015749	0.126943330	1.164188
3	0.58113883	0.295430315	0.273661733	1.150231
4	0.14707867	0.609148974	0.400414707	1.156642
5	3.08248290	0.013700333	0.010257752	3.106441
6	0.00000000	2.684031499	0.015589113	2.699621
7	0.96116135	0.126247880	0.118033989	1.205443
8	0.01015254	1.924017738	0.032474494	1.966645
9	1.35702260	0.068387297	0.064085069	1.489495
10	0.17851130	0.656503812	0.329573974	1.164589
11	0.88982237	0.137183010	0.145890190	1.172896
12	0.54303350	0.314767947	0.309574683	1.167376
13	0.21267813	0.259921050	0.699044245	1.171643
14	0.07832773	1.027400665	0.109466047	1.215194
15	0.29099445	0.228010500	0.634812656	1.153818
16	0.71498585	0.277182387	0.166545171	1.158713
17	1.88675135	0.035744169	0.038425603	1.960921
18	0.31306433	0.380613560	0.460139433	1.153817
19	1.67261242	0.043532011	0.044552273	1.760697
20	0.82574186	0.185631101	0.155987115	1.167360
21	0.03142125	1.554364775	0.050864054	1.636650
22	0.44337567	0.335314515	0.374708102	1.153398
23	0.13227703	0.771097615	0.257433430	1.160808
24	0.05409255	0.086346741	1.236067977	1.376507
25	4.00000000	0.006756962	0.005063453	4.011820
26	0.02062073	0.051559496	1.659147948	1.731328
27	0.62221421	0.357208808	0.189207115	1.168630
28	0.36082763	0.213213752	0.581138830	1.155180
29	0.41421356	0.528553544	0.201405707	1.144173
30	0.23091491	0.148555594	0.778279410	1.157750
31	0.47441956	0.462008869	0.242188995	1.178617
32	1.04124145	0.105578099	0.101216657	1.248036
33	0.33630621	0.405721109	0.428720215	1.170748
34	1.13200716	0.095793708	0.085592604	1.313393
35	0.66666667	0.199105336	0.290994449	1.156766
36	2.53553391	0.020839303	0.021064185	2.577437
37	0.11803399	0.160397208	0.880301547	1.158733
38	0.50755672	0.432760812	0.227826015	1.168144
39	0.09108945	0.115721583	1.020515505	1.227327
40	0.04257207	1.320794417	0.071017586	1.434384
41	0.27000127	0.709975947	0.177603707	1.157581
42	1.23606798	0.077217345	0.078181983	1.391467
43	0.76776695	0.172742786	0.214256794	1.154767
44	0.16247639	0.493801582	0.495348781	1.151627
45	2.16227766	0.028183723	0.026690096	2.217151
46	0.38675049	0.243556587	0.535259784	1.165567
47	1.50000000	0.059839833	0.057371263	1.617211
48	0.25000000	0.566783120	0.351200155	1.167983
49	6.07106781	0.000000000	0.000000000	6.071068
50	0.10431526	0.925856786	0.136219366	1.166391
Σ	40.17290093	21.913554642	15.003622405	

(C)	1	2	3	∏
1	0.68	0.62	0.16	0.067456
2	0.04	0.94	0.96	0.036096
3	0.48	0.48	0.30	0.069120
4	1.00	0.98	0.02	0.019600
5	0.96	0.04	0.94	0.036096
6	0.78	0.10	0.84	0.065520
7	0.26	0.38	0.68	0.067184
8	0.34	0.40	0.50	0.068000
9	0.06	0.92	0.90	0.049680
10	0.64	0.30	0.36	0.069120
11	0.82	0.82	0.10	0.067240
12	0.08	0.88	0.86	0.060544
13	0.90	0.06	0.92	0.049680
14	0.62	0.46	0.24	0.068448
15	0.46	0.24	0.62	0.068448
16	0.72	0.16	0.60	0.069120
17	0.42	0.28	0.58	0.068208
18	0.60	0.80	0.14	0.067200
19	0.74	0.36	0.26	0.069264
20	0.52	0.18	0.72	0.067392
21	0.58	0.42	0.28	0.068208
22	0.20	0.44	0.78	0.068640
23	0.98	0.02	1.00	0.019600
24	0.50	0.76	0.18	0.068400
25	0.94	0.96	0.04	0.036096
26	0.02	1.00	0.98	0.019600
27	0.66	0.14	0.74	0.068376
28	0.12	0.74	0.76	0.067488
29	0.32	0.54	0.40	0.069120
30	0.88	0.86	0.08	0.060544
31	0.14	0.72	0.66	0.066528
32	0.36	0.60	0.32	0.069120
33	0.18	0.70	0.54	0.068040
34	0.70	0.12	0.82	0.068880
35	0.40	0.50	0.34	0.068000
36	0.10	0.84	0.80	0.067200
37	0.22	0.64	0.48	0.067584
38	0.84	0.68	0.12	0.068544
39	0.44	0.78	0.20	0.068640
40	0.38	0.34	0.52	0.067184
41	0.86	0.08	0.88	0.060544
42	0.80	0.22	0.38	0.066880
43	0.54	0.56	0.22	0.066528
44	0.28	0.58	0.42	0.068208
45	0.16	0.66	0.64	0.067584
46	0.76	0.20	0.44	0.066880
47	0.56	0.26	0.46	0.066976
48	0.92	0.90	0.06	0.049680
49	0.24	0.52	0.56	0.069888
50	0.30	0.32	0.70	0.067200
Σ	25.50	25.50	25.50	

Table 5. (A): the matrix $\tilde{\boldsymbol{X}} \in O_+(\underline{\boldsymbol{X}})$ approximating for $n = 50$ the minimal expectation of $d = 3$ homogeneous Exp(1)-marginals; (B): the matrix $\tilde{\boldsymbol{Y}} \in O_+(\underline{\boldsymbol{X}})$ approximating for $n = 50$ the miminal expectation of the stop-loss function for the inhomogeneous Pareto portfolio underlying Table 4; (C): the matrix $\tilde{\boldsymbol{Z}} \in O_{\psi}(\underline{\boldsymbol{X}})$ approximating for $n = 50$ the miminal expectation of the product of the three uniform marginals underlying Table 1.