Using Medians in Portfolio Optimization

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Abstract: This paper formulates a number of new portfolio optimization models by adopting the sample median instead of the sample mean as the efficiency measure. The reasoning behind this is that the median is a robust statistic, which is less affected by outliers than the mean. In portfolio models this is particularly relevant as data is often characterized by attributes such as skewness, fat tails and jumps that are incompatible with the normality assumption. Here, we demonstrate that median portfolio models have a greater level of diversification than mean portfolios, and that, when tested on real financial data, they give better results in terms of risk calculation and concrete profits.

Keywords: Finance, risk management, portfolio models, mixed-integer linear programming, robust statistics.

1 Introduction

The Markovitz model is founded on the assumption that when making financial decisions, investors should always seek to balance return and risk. And it is the need to translate this premise into an operational tool that has led to the classic mean variance optimization model that is the milestone of current portfolio theory. The mean variance model assumes that returns follow a multivariate normal distribution and, if an investor is fully aware of the mean variance and co-variance parameters, then he or she can select a point on the efficient frontier, [34]. Of course, we can never be completely sure of market parameters, as they have to be estimated from available data.

The standard approach is to use maximum likelihood estimators (the sample mean and the sample variance and covariance), and then to formulate a quadratic optimization problem, see chapter 8 in [17].

Any assumption of normality, however, is decidedly questionable, as there are many examples of market data that do not follow a normal distribution pattern, as they are subject to fat tails, abnormal jumps, skewness and so on, that are incompatible with normality, [41]. This poses the inevitable question of whether maximum likelihood estimators really are the best way of approaching this problem. Without doubt, they are the most efficient approach in situations where data is normal, but what happens when data is not normal? And at a more detailed level, is it safe, when dealing with financial data, to consider efficiency, e.g. the property that an estimator has minimum variance, as the estimator's most prominent feature?

One way to address the question of data non-normality is to use robust statistics for the means and variances, or as is more generally stated, for what are called the distribution location and scale parameters. Robust statistics were developed almost 50 years ago to deal with the problem of making correct estimations when data is only approximately normal. Initially, non-normality was described in terms of contaminated distributions, e.g. a mixture of a two normal distributions, where one is responsible for most of the data, but the other disturbs the process by adding a a number of outliers. Surprisingly, this decreases the efficiency of maximum likelihood considerably and just 5% of contaminated data is enough to turn robust estimators, as trimmed means and the median, as efficient as the sample mean, [44].

Contaminated random variables are a compelling hypothesis for portfolio optimization. After all, on many trading days data is regular, and so no one feels the need to question normality. Non-normality is evident, however, when outliers, or jumps appear, but this happens rarely and we don't have enough data to ascertain their proper distribution. So, robust statistics may well be the most appropriate tools, as they don't assume that data are given by any peculiar random variable. Rather, the remarkable property of robust statistics is that they work reasonably well under any circumstances whatsoever.

If we accept that all the considerations made so far in this paper make sense, then there is clearly room for exploring the possibility of using the median in place of the mean as the estimator of portfolio efficiency. And, if this is true, it follows that this estimator can then be used to formulate a new portfolio optimization model that continues to adopt Markovitz's risk return approach. Our approach is, therefore, really quite straightforward, we have simply replaced the mean in this model with the median. At first, this may look as if we have done very little beyond discarding a linear function (the easiest to optimize), and replacing it with a difficult order statistic to create an APX-complete problem, [10], but let's see what the advantages are.

In terms of asset location, the most important of the two Markovitz measures, mean and variance, is the mean, [13, 16]. Therefore, any misspecification of the mean will have considerably greater consequences than mistakes regarding variances and co-variances. It is also argued by some authors, [28], that the sample mean data parameter is too sensitive to be reliable and is therefore not suitable for portfolio optimization. This assertion has two consequenses. First, it sustains the optimality or near-optimality of passive investment strategies like the $1/N$-portfolio, e.g. equal weights on all assets, [18]. Secondly, it has pushed some authors to discard the mean altogether from the optimal portfolio model and to calculate the optimal decisions using the simple minimum variance model, [19]. On the other hand, this assumption that the mean is of little use contrasts strongly with other experiments focusing on momentum strategies, in which the sample mean plays an important role. A momentum strategy allocates wealth to assets that have registered the highest mean performance in the past, and this has proven [5, 29, 42] to result in effective decisions. So when all is said and done, we believe that between the extremes of accepting and discarding the sample mean entirely, there is room for experimenting with different location and efficiency estimators, such as the median and other robust statistics.

This is not the first time robust statistics have been used in a portfolio optimization framework, but previous works have tended to focus more on robust variance or scale estimators rather than on questioning the use of the mean. In [19], a minimum variance portfolio model is proposed, in which the sample variance is replaced by an M-estimator or an S-estimator. From a computational point of view, this involves solving a non-linear programming problem where the variance has to be estimated and optimized in a single step. This one step approach compares favourably with other approaches where estimations and optimization are carried out in two separate stages, e.g. a robust estimation of the mean and variance is calculated first and then the results are used to solve a standard mean/variance quadratic programming problem. This approach has been used, for example, by [22, 31, 35, 45]. Other studies also include ways of addressing the estimation error. The term 'robust', for example, has been used for models, which recognize that parameter variability can be caused by the estimation process and confidence levels, but still consider the data to be within the framework of normality. In this case, so-called robust optimization assumes a max-min approach, in which the decision-maker optimizes a worst-case scenario, see [40] and the references therein.

In this paper, we present a number of new portfolio optimization models, where the portfolio mean is replaced by the median. The core problem is to maximize the median, which can be formulated by finding the convex combination of arrays, e.g. the portfolio weights, such that the resulting vector, e.g. past portfolio returns, has the maximum median value. In [10], we prove that the Optimal Median is a NP-complete problem, and that it can be formulated as a Mixed Integer Linear model. There is an interesting financial consequence to this finding, despite its inherent difficulties. When the median objective function is plotted in a bidimensional space, it transpires that the optimal median can be an internal point of simplex, and this means that optimal median portolios can consist of many assets, all with weights of less than 1. This is contrasts sharply with presumable Optimal Mean portfolios, in which the objective function is linear, where optimal portfolios can only consist of one single asset with weight 1. This implies that the Median model has the appealing property of introducing portfolio diversification, even when *no risk measure is introduced to complete the model.* The computational aspects of the Median problem are also developed in [8, 26].

This paper is structured as follows. The second section contains the formulation and the main properties of the Optimal median problem, where we will show how the two processes of parameter estimation and optimization are carried out in a single step. In the third section of the paper, we create a fully-fledged risk/return model by adding a number of risk measures to the problem. Please note that, since the optimal median is itself a MILP model, only risk measures formulated as linear constraints are used, like the ones experimented with in [33]. We have experimented with 4 different risk measures: Mean Absolute Deviation [27], Value-at-Risk [30], Conditional Value-at-Risk [38], and Maximum Loss, [46], and we have ruled out quadratic risk measures, such as sample variance and its variations. This is because in our financial applications, as will be seen, the sample variance/co-variance is a singular matrix, and therefore unusable in a quadratic optimization model.

The fourth section of the paper describes the main financial application of our new models, i.e. the allocation of assets in the Milan stock market. Over the last few years, we have worked together with a small Italian bank, providing their traders with computer programs to implement optimal portfolio models, as documented in [11]. In this period, we have observed that the portfolio models are sometimes cheated by assets where the sample mean is high, simply because their data contains an isolated jump in a context of flat returns. The models have nevertheless selected these asset, but their perfomance has often been unsatisfactory, because the jumps are not replicated. Isolated jumps, however, are outliers in statistical terms, of course, and statisticians deal with them using robust statistics. That is how we first began to think of using the median as an objective portfolio function. As we will see, the results we have achieved by applying the median/risk model to the Milan stock market are very encouraging, as they are the best portfolio models in our experiment. Finally, in the last section we test the robustness of the models by applying them to 13 different data sets taken from [6, 14, 15].

If we were to summarize all our results in single sentence, then our overall claim is that median models calculate portfolios with a higher asset diversification than mean models, and this, I think it is fair to say, is a significant financial breakthrough.

2 The Optimal Median Problem

Let R be a random variable whose distribution is unknown, and from which a sample $r = \{r_1, r_2, \ldots, r_T\}$ is drawn. Rank r in a non-decreasing order to obtain the ordered array:

$$r_{1:T} \leq r_{2:T} \leq \ldots \leq r_{T:T},$$

so that $r_{i:T}$ is the value in position i on a list of T. For the sake of simplicity, let T be odd, so there is no ambiguity regarding the middle value of r. The estimated median of R is $med[R] = r_{\frac{T+1}{2}:T}$.

The sample median is popular for being a robust location estimator of a distribution function. Its optimality properties are described in terms of the breakdown point, [20], and the influence function, [23]. The breakdown point of an estimator is the smallest fraction of observation that has to be replaced to make the estimator unbounded[1]. The breakdown point of the sample median is 50%, as at least half of the data have to be replaced to push the sample median outside the range of the original data, while the breakdown point of the sample mean is $\frac{1}{T}$. The influence function is the measure of how an estimator, say $\hat{\mu}$, at a distribution of F is affected by contaminating the F with an outlier datum. The influence of contamination on the median is bounded by the constant $\frac{1}{f(F^{-1}(1/2))}$, while the influence on the sample mean is unbounded. That is, any contamination, however small, but sufficiently far from the estimated mean, can carry the mean arbitrarily far away from its initial value. In the context of financial data, it can be the case that the mean is affected by critical factors resulting from its weakness with regard to these properties.

[1]Replaced data and outliers magnitude can be chosen in the least favourable way.

In the Optimal Median Problem, the returns of K assets are modeled as multivariate random variables $R_1, \ldots, R_K$, from which we can take samples of R_j in the form of time series $r_j = \{r_{1j}, \ldots, r_{Tj}\}$. Then all the data is collected in a matrix $R \in \Re^{T \times K}$, in which matrix entry r_{ij} is the return on the time i of the asset j. Let x_j be the percentage of wealth allocated on asset j, so the portfolio is a random variable Z that is linked to R_j through the linear formula $Z = \sum_{j=1}^{K} x_j R_j$, and the Z time series is $z_t = \sum_{j=1}^{K} x_j r_{jt}$.

To formulate the Optimal Median problem, assume that the investor cannot go short and that all wealth must be allocated, then problem **P1** is:

$$\max_{x,z} \left(z_{\frac{T+1}{2}:T} \right)$$

s.t.

$$z_i = \sum_{j=1}^{K} x_j r_{ij} \text{ for } i = 1, ..., T \tag{1}$$

$$\sum_{j=1}^{K} x_j = 1 \tag{2}$$

$$x_j \geq 0 \text{ for } j = 1, ..., K \tag{3}$$

As can be seen in [10] and in figure 1, the median of a convex combination of vectors is a non-differentiable piecewise linear function. It is characterized by many local optima, in which the function is non-differentiable. From an optimization point of view, [10] proves that the Optimal Median Problem is APX-complete, that is, the problem cannot be solved, even if a fixed tolerance ϵ from an optimal solution is allowed. It follows that optimal solutions to the problem can only be calculated with exponential algorithms, or by implementing heuristic procedures to obtain a fast solution.

The following reformulation puts the Optimal Median in the form of a Mixed-Integer Linear Programming model. Let z^{Med} be the variable representing the median; let $y_i, i = 1, ..., T$ be the binary variables representing the inequalities that define z^{Med}, let M be a sufficiently large number. The resulting problem, referred to as **P2**, is :

$$
\begin{aligned}
&\max_{x,y,z^{Med}} z^{Med} \\
&s.t. \\
&z^{Med} \leq \sum_{j=1}^{K} x_j r_{ij} + M(1 - y_i) \quad \text{for every } i = 1, \ldots, T & (4) \\
&\sum_{i=1}^{T} y_i = \frac{T+1}{2} & (5) \\
&\sum_{j=1}^{K} x_j = 1 \\
&y_i \in \{0, 1\} \quad \text{for every } i = 1, \ldots, T \\
&x_j \geq 0 \quad \text{for every } j = 1, \ldots, K.
\end{aligned}
$$

The constraints (4) are active iff $y_i = 1$, otherwise they are not binding; z^{Med} is the median of z if there are $\frac{T+1}{2}$ z_i values that are greater or equal to z^{Med}, therefore variables y_i, constrained by (5), define which linear equations characterize the local optima of figure 1.

From a financial point of view, the most remarkable consequence of non-convexity is that the optimal median portfolio, e.g. the solution x, is not restricted to being an extreme point of the simplex $S = \{x \geq 0, \sum_{j=1}^{K} x_j = 1\}$, as it would be if we calculated the Optimal Mean, but instead, it can be an internal point of the $0 \leq x \leq 1$ range. That is to say, if we use the Optimal Median to make financial decisions, we can obtain optimal portfolios containing more than one asset.

Compare this case with the classic Mean/Variance approach where only the risk measure, e.g. the variance, to brings diversification to the portfolio. If we use the Mean/Variance without the Variance, then the portfolio we end up with will have one asset only!

Problem **P2** is not a method for making portfolio decisions because it contains no risk index. What interests us here is how much diversification it can bring to the portfolio and how it can be useful to avoid bad investments. To achieve this, in the simulation studies it will be benchmarked against the mean-variance model formulated without the variance constraint, i.e. as a straight Best Mean Problem **P3**:

$$\max_x \frac{1}{T}\sum_{i=1}^{T} z_i$$

$$z_i = \sum_{j=1}^{K} x_j r_{ij}$$

$$\sum_{j=1}^{K} x_j = 1$$

$$x_j \geq 0 \quad \text{for every } j = 1, \ldots, K.$$

The Best Mean solution is $x_j = 1$ if j is the asset with the best sample mean, or $x_i = 0$ otherwise. As we have already stated, the Optimal Mean does not induce any diversification and also imposes the maximum risk measure on the portfolio. As we will see in the computational section, in certain market conditions this property can lead to high losses, which can be reduced using the Optimal Median model.

3 Introducing Risk Measures

The optimal median model presented in the previous section can be further enhanced using a risk measure, and there are numerous studies from which this can be taken. Unfortunately, variance is not an option, because of the singularity of the matrix. It is easy to see that, using sample data collected in the matrix R, the sample co-variance matrix is calculated as:

$$\Sigma = \frac{1}{T} R(\mathbf{I} - \frac{1}{T}\mathbf{1}\mathbf{1}^t)R^t$$

which is an idempotent matrix. As such, $Rank(\Sigma) = Rank(\mathbf{I} - \frac{1}{T}\mathbf{1}\mathbf{1}^t) = T - 1$. In practice, the condition $T < K$ is frequent. For example, an investor wishes to optimize over all stocks traded on a particular market, and therefore K can be more than one hundred. As a second example, T can be a small time window, for example, because the economy is at a turning point (between recession and recovery, for example), that makes older data obsolete. In this case, T are ten or twenty at most, and $T << K$. In any case, the singularity of the matrix makes any model that uses the sample variance inconsistent.

Furthermore, the Optimal Median is calculated by any software that can solve a MILP problem. So when risk measures are added, the problem should preferably not be made more complex, but remain in the MILP class, in order to fully exploit existing technology as this will save researchers from having to create their own software. Even with these restrictions (e.g. the fact that risk measures should be represented by linear constraints, but still allow for binary variables), there are many risk measures that can be used. We experimented with the following four:

- the Mean Absolute Deviation (MAD), the resulting portfolio model will be called MedMAD;
- the Value-at-Risk (VaR), the resulting model will be MedVaR;
- the Conditional Value-at-Risk(CVaR), the resulting model is MedCVaR;
- the Maximum Drawdown, (Max), giving the MedMax portfolio model.

The Optimal Median is then paired with any of the four measures to obtain four different optimization models.

The Median Absolute Deviation is presented in [27] and is calculated as follows. Suppose a parameter t is given, then the MAD from t of a return vector z (regarded as occurrences of the rv Z) is:

$$MAD_t[Z] = \frac{1}{T}\sum_{i=1}^{T} |z_i - t|$$

According to existing literature, there are two alternatives for t, e.g. the sample mean or the median of Z. The original paper [27] uses $t = E[Z]$, but in our application, the estimation $med[Z] = z^{Med}$ is available. In line with our model assumptions, we use $t = z^{Med}$ and then:

$$MAD[Z] = \frac{1}{T}\sum_{i=1}^{T} |z_i - z^{Med}|$$

If data follows a multivariate normal distribution, then two scenarios may occur. If the parameters are known, then theorem 1 of [27] applies, by which the Mean/Variance and the MedMAD calculates the same efficient frontier[2]. Of course, this is not the case with real applications, where parameters must

[2]If parameters are known, then $\mu = E[Z] = med[Z]$ and $Var[Z] = kMAD[Z]$, with k being the appropriate constant.

be estimated. In this case, and if data is normal, then the MAD is less efficient than the mean, as documented in [43]. But when data is contaminated, the properties of the MAD are appealing, see [44], as just 10% of contaminated data is enough to make the MAD estimators more efficient than the sample variance. The MAD is sometimes erroneously referred to as a robust estimator, but it is not, as its breakdown point is 0 and its influence function is unbounded, [39].

The MAD is inserted as a risk measure in problem **P2**, using the following formulation. First, to control for the variability maximum, a threshold of v^* must be established. Then, these inequalities must be added to the Optimal Median Problem:

$$v_i \geq z_i - z^{Med} \text{ for all } i = 1, \ldots, T \tag{6}$$

$$v_i \geq z^{Med} - z_i \text{ for all } i = 1, \ldots, T \tag{7}$$

$$\frac{1}{T}\sum_{i=1}^{T} v_i \leq v^*$$

The constraints (6) and (7) stand for the non-linear $v_i \geq |z_i - z^{Med}|$ which together with the property $\sum_i |z_i - z^{Med}| = \min_t \sum_i |z_i - t|$ make the model formulation correct.

Between the four models we experimented with, the MedMAD model is the closest to the original Markovitz formulation. The mean/variance Markovitz formulation was viewed as a location/scale trade-off problem regarding a random variable Z. Then the problem was reformulated using the sample Median as the location estimator of Z, and the sample MAD as the scale estimator of Z.

The MedMAD model is the only new model that uses a symmetric index as a risk estimator. The other three models used indexes calculated on the tail of the distribution as risk measures. This decision was based on the observation, supported by existing literature, that risk should be connected to the bad occurrences of a random variable, not to positive or negative returns.

The first tail measure we used was Value-at-Risk, [30]. The definition of Value-at-Risk coincides with that of the α-quantile of a distribution F. Let Z be the random variable that describes the portfolio return and fix the probability threshold as $\alpha \in [0, 1]$, so the Value-at-Risk is:

$$VaR_\alpha[Z] = -\min\{t | \Pr[Z \leq t] \geq \alpha\}.$$

Note that in this definition, positive VaR stands for losses. To introduce Value-at-Risk in an optimization model, first one must observe that the decision-maker can describe the distribution of Z with its historic achievements, i.e. its set of portfolio returns $z_1, z_2, \ldots, z_T$. Then, the distribution of Z is approximated by $\Pr[z = z_i] = p_i = \frac{1}{T}$ and a new variable, $-z^{VaR}$, standing for the Value-at-Risk quantile, must be introduced. Let z^* be the greatest Value-at-Risk that is accepted by the investor, introduce new binary variables w_i, and the model MedVaR is obtained by adding the following constraints to **P2**:

$$-z^{VaR} \leq \sum_{j=1}^{K} x_j r_{ij} + M(1 - w_i) \text{ for every } i = 1, \ldots, T \tag{8}$$

$$z^{VaR} \leq z^* \tag{9}$$

$$\sum_{i=1}^{T} z_i = T \lceil (1 - \alpha) \rceil \tag{10}$$

$$w_i \geq y_i \text{ for every } i = 1, \ldots, T \tag{11}$$

$$w_i \in \{0, 1\} \quad \text{for every } i = 1, \ldots, T$$

These constraints are discussed in detail in [9], which shows that the resulting problem is NP-complete, but if T is a fixed parameter, then polynomial algorithms are possible, and a global optimization approach for VaR optimization is also proposed in [21]. Here, the constraints (8) and (10) are the same as the constraints defining z^{Med} as both the variables z^{Med} and z^{VaR} are defining the quantile of a distribution. The constraints (11) connect the binary variables that define the median with the one defining the Value-at-Risk: more precisely, if an inequality of the form $z^{Med} \leq \sum_{j=1}^{K} x_j r_{ij}$ defines the value of the median, then it also defines $-z^{VaR}$ through the inequality $-z^{VaR} \leq z^{Med} \leq \sum_{j=1}^{K} x_j r_{ij}$. As can be seen, the model requires the introduction of T new binary variables for a whole of $2T$. However this size is not critical and the computational times that we experimented are no longer than those of other models. Last, (10) is the constraint on the maximum VaR.

At the time VaR was introduced for financial decisions, many researchers were criticizing the use of Value-at-Risk, on the basis that if a loss exceeds the Value-at-Risk, then it can be any size whatsoever. In mathematical terms, it was soon realized that Value-at-Risk lacks the subaddittive property, i.e. a property that is fundamental for controlling risk as it imposes diversification on a mathematical basis. This is the main assumption of the so-called Coherent Measures of Risk, [4, 1]. The correction that was proposed is straightforward and is called Conditional Value-at-Risk, e.g. CVaR in [38], or Expected Shortfall [2], i.e. the expectation of shortfall due to unfavourable conditions.

To calculate CVaRs, first the Value-at-Risk has to be calculated, then the expectation of the random variable Z is computed on the condition that it takes values below the VaR. As before, the occurrences of $z_1, z_2, \dots, z_T$ are available, and the distribution of Z is approximated using $\Pr[z = z_i] = p_i = \frac{1}{T}$, then:

$$CVaR_\alpha[Z] = \frac{1}{1-\alpha} \sum_{i:-z_i \geq VaR_\alpha[Z]} p_i(-z_i)$$

As is shown in [38], which is true for any coherent risk measure as they can all be easily linearized, see [7], the CVaR can be easily inserted in portfolio models. First, introduce the continuous variables $-z^{VaR}, v_i$ and the parameter z^* for the highest CVaR tolerable, and then insert the following constraints:

$$-z^{VaR} + \frac{1}{(1-\alpha)T} \sum_{i=1}^{T} v_i \leq z^*$$
$$v_i \geq 0 \text{ for every } i = 1, \dots, T$$
$$v_i \geq -r_i + z^{VaR} \text{ for every } i = 1, \dots, T$$

As all new variables are continuous, the model is easy to optimize. It is also one of the most commonly used models in financial contexts where variance is replaced by a tail risk measure, such as in a credit risk model, [3].

The last measure we considered was the Max Drawdown (MaxD), that was introduced in [46]. It is defined as $z^{Min} = \min\{z_i | i = 1, \dots, T\}$. Then, letting z^* be the maximum acceptable loss, the constraints to add are:

$$z_i = \sum j = 1^K x_{ij} r_{ij} \text{ for all } i = 1, \dots, T$$
$$z_i \geq z^* \text{ for all } i = 1, \dots, T$$

The quantity z^{Min} is a quantile estimator, [24], and has been recently extended in [36].

We therefore ended up with 4 variants of our Median/Risk portfolio model. All the models are ILP, but the MedCVaR and MedMax are the ones requiring less constraints and variables, while the most difficult to solve was the MedVaR model, as it introduces T new binary variables. In our computational tests, the problems were all small-sized and the computational burden was not an issue. However, if the problem were to become significantly larger , then these differences should be taken into account.

4 Test results

We used these new models to simulate an investor who utilises the last T observations to calculate an optimal portfolio, and then waits for W new observations, (e.g. the rolling window W), before optimizing the portfolio again. The new portfolio models are tested with these specifications of α[3]:

- The Best Median with no risk constraint.
- The Best Median with Median Absolute Deviation constraint.
- The Best Median with Value-at-Risk constraint, with $\alpha = 0.25$.
- The Best Median with Conditional Value-at-Risk constraint, with $\alpha = 0.10$.
- The Best Median with Maximum Drawdown constraint.

In order to be as objective as possible, we used the following method to specify the constraints on the maximum tolerable risk. All the models take as their input a $T \times K$ matrix of returns. Then the market index is calculated assuming that wealth is allocated evenly on all assets, e.g. using the $\frac{1}{K}$ rule [18]. The result is a time series of T data, on which the VaR, CVaR, MaxD and MAD of the 1/K portfolio index can be calculated. These values are used as the risk benchmarks for optimization in the next stage. In other words, the decision maker calculates a new median/risk portfolio, on the condition that the expected risk is less than the risk observed in the last stage on the $1/K$ portfolio. The portfolio weights are kept constant until a new optimal portfolio is calculated[4].

To analyze the performance and other financial features of median portfolios, we have also compared our five models with two benchmark portfolios:

- The Equal Weight Portfolio (EqW), in which wealth is evenly allocated between all assets (implementing the $1/K$-strategy).

[3]The value of α was chosen as reasonable for the problem in hand, and no attempts were made to find the best one.

[4]This implies rebalancing on every t.

- The Best Mean Portfolio, in which the problem **P3** is solved and all wealth is allocated to the asset with the best sample mean.

These benchmarks represent the two most extreme strategies of a fund manager. Equal weight is a passive strategy [5] that does not use any of the available information. Here, portfolio diversification is as high as it can get, which most likely implies a very low ex-post risk. It is important to note that the $1/K$ portfolio is a very demanding benchmark, more demanding than an official index for example, as [18] proves that when a number of portfolio strategies are compared, this one is by far the best. Furthermore, uniform portfolios provide an answer to one of the basic financial questions that matter to investors: is any advantage to be gained by using an active strategy instead of a passive one? Clearly, if new models did not obtain a higher return than the EqW, then they are pointless, as it is almost impossible for them to achieve less risk.

The second benchmark portfolio is the Best Mean. Clearly, it is a paroxistic and unrealistic method of investing, but not as airy-fairy as it seems. First of all, the model can be seen as an application of the so called 'momentum strategies', in which an investor simply has to follow the market and pick up assets with the best past performance. Those strategies are tested in [5, 29, 42] and the results were surprisingly good. Secondly, some authors claim that sample expectation is such an unreliable estimator that it can be discarded completely from portfolio models, [28]. But this is an issue that needs to be examined in greater detail, which is what we have done. Last but not least, it is also important to remember that the Best Mean is the best benchmark for the Best Median model, as both models lack a measure of risk.

All these models have been applied to decision making situations using real financial data. The first section regards experiments made on daily data from the Milan stock market. These experiments used specific portfolio models from the trading desk of a small Italian bank that we frequently work with, [11]. The second section describes how we applied the models to return data taken from current literature, or to be more specific, to data taken tests carried out by Beasley *et al.*, [6, 14], and Cesarone *et al.*, [15]. The main differences between the two experiments is that certain specifications in the Milan stock market experiment, e.g. daily versus weekly data, monthly rolling windows, and so on, were prompted by current financial investment practices. When the same parameters were applied to Beasley and Cesarone's public data, a certain degree of realism was inevitably lost, but at least we

[5]The strategy is passive in the sense that it does not use any market information. However, the $1/K$ ratio rebalances the portfolio at every stage.

did not bias the experiments by finding the best ways of implementating the new models.

For every test in which the portolio models were applied, an ex-post portfolio return time series was obtained showing the gains or losses made by the investor. We also registered the relevant means and distribution quartiles for each time series. So, when the models are compared in terms of highest returns, the highest mean stands for the wealthiest portfolio[6]. When we compare risks, on the other hand, we refer to the time series maximum loss. Standard deviations are not reported, as the return distributions show asymmetric gains and losses and we did not want to penalize gains[7].

The models were also used to calculate the portfolio weight vector. Weights give the portfolio diversification objectively suggested by the models. It is important to note that this diversification is not imposed by the investor, using upper bounds on weights for outside contingent information. To compare diversification we used the following two indexes: the Herfindahl-Hirschman (HH) Index [25] and the Max index (Max), (i.e. the maximum portfolio weights).

In each period t, the models calculate the portfolio weights $x_{t1}, \dots, x_{ti}, \dots, x_{tK}$, such that $x_{ti} \geq 0$ and $\sum_{i=1}^{K} x_{ti} = 1$, therefore, the HH-index, that is commonly used for market concentration, [25], is a suitable concentration/diversification index. It is:

$$HH_t(x) = \sum_{i=1}^{N} x_{ti}^2.$$

The second index, the MAX, reports the greatest portfolio weight in each period t and is calculated as:

$$MAX_t(x) = \max\{x_{ti} | i = 1, \dots, N\}$$

So, after applying the models, we obtained the time series of the indexes and registered the mean values of both.

Note that the two benchmark portfolios obtain the upper and lower bounds of the two indexes: the $1/K$-portfolio gives weights x such that $HH_t(x) = MAX_t(x) = 1/K$, the Best Mean model calculates x such that $HH_t(x) = MAX_t(x) = 1$.

All tests were carried out in R, [37], using the lpSolve and lpSolveAPI packages, [12, 32]. We will try to publish our subroutines too, if it is possible.

[6]One can freely compare return medians, but the result does not change much.

[7]The asymmetry of many distributions is evidenced by the fact that the means and medians obtained are often very different.

4.1 The Milan stock market

The Milan stock market data is made up of returns calculated from the daily market prices of 60 companies, who traded throughout the time period ranging from March 2003 to March 2008. The Milan stock market is the main market in which the small Italian bank, the Cassa Rurale di Rovereto, invests, and their investment practice and the way in which they apply portfolio models has been documented elsewhere, [11]. To give an idea of the effectiveness of their decision making, figure 2 shows a comparison between the Equal Weight portfolio, which approximates a passive strategy and stands for a market index, with the Mean/CVaR portfolio model, [38], which is one of the Bank's favourite models and approximates the Bank's active decision making. The MeanCVaR model is specified with $\alpha = 0.1$, $T = 50$ and no-shortselling, weights are recalculated for every $W = 20$ new data, the CVaR threshold is the CVaR of the equal weight portfolio of the last T days. These parameters reflect the views of the bank traders, as they trust daily prices more than weekly prices. Moreover, they only consider relevant returns from the last T=50 trading days, e.g. data from the last two and a half months, as markets change direction frequently and older data soon becomes obsolete. As can be seen from the figure, during the bull market before the 2007 crisis, the CVaR model provided a stable wealth growth, and obtaining better results than the ones using the EqW portfolio, as the CVaR portfolio is 50% more than the EqW (see also table 1).

On the basis of the widespread practice established in these pre-crisis years, and convinced that porfolio models are useful tools for decision making, bank practitioners have suggested two ways in which portfolio models could be upgraded.

- Models should distinguish stocks whose high expectation, calculated from the sample data, is due to stable growth, from stocks in which a few isolated jumps determine a high sample mean.
- Models with short time series should be made available, i.e. with $T << 50$, as sometimes the announcement of a new economic policy changes the economic scenario and makes old data obsolete and unreliable.

But what exactly is the contemporary take on these issues? Regarding point 1, our bank contacts pointed out that price time series are characterized by jumps, for example, following the publication of positive business news, e.g. when a company signs a new contract or officializes a new alliance. The most typical - and unreliable - jump is when a takeover bid is made for a company. For example, figure 3 shows a price graph for the Italian Bulgari company, in the early months of 2011. As can be seen, the price line begins with an irregular trajectory of small price adjustments, before making a price jump of almost 50% following a takeover bid by the French group LVH. After the jump, however, prices remained in a close range around the bid price. In the following months, all the mean portfolio models selected Bulgari as a main asset, but this is clearly a mistake, as the high level of recent gains could clearly not be repeated in the future.

With regard to the second issue, demanding short time series models (of $T = 20$ or even less), the economic rationale is clear. Markets are driven by economic policies made by many interdependent decision-makers, whose behaviour is impossible to predict. Who knows how the European sovereign debt crisis will evolve, for example? A completely new scenario may emerge after even just one solitary policy decision, like a quantitative easing agreement, for example. In any radically new market condition, the data that can be used to forecast prices will be limited to just a few days, as any older information will no longer be reliable. Unfortunately, portfolio models are not built for use with short time series, and even the strongest arguments favouring the sample mean inevitably rely on the large number law, which assume long time series.

We have addressed both points by replacing the mean, as the measure of portfolio efficiency, with the median. As far as point 1 is concerned, the median, being a robust statistic, is not affected by jumps that are treated as time series outliers. Regarding point 2, it is common knowledge that the highest level of statistic efficiency (e.g. least variance) when using the mean as a location parameter estimator, is only obtained on the condition that data is normal *and* a sufficient long time series is available. In practice, when one of these conditions, or both, are not met, then there is a valid reason for experimenting with other location estimators.

For this reason, we compared the first two models: the BestMean with the BestMedian model. Neither of them include a risk measure, so we observed the pure buying signal issued by the two measuers. We also tried to establish how significant the hypothetical diversification effect of the Median, caused by the presence of so many local optima, was. For both models, and for all the median/risk models, we used $T = 21$, an odd number to avoid any ambiguity when defining the median.

If we look at figure 4, one of the drawbacks of the BestMean model becomes apparent. The mean is an efficient measure that cannot diversify the portfolio, and even worse, as it invests completely in one asset only, it obtains maximum concentration and maximum risk. As a consequence, extreme returns in terms of both gains and losses are possible. The test shows that with the Milan Stock Market data, for many periods the Best Mean portfolio mimiced EqW index returns, but with a higher level of volatility. This strategy is extremely risky as it traps investments in one asset only, e.g. Impregilo, whose price plummeted at the end of 2004. As a result, the final wealth of the BestMean portfolio is much less than the EqW[8].

The Median model avoids crashes like these. Figure 5, plots the Best Median model returns. The dips and volatility that characterize and affect the Best Mean returns are nowhere to be seen, and, even more importantly, the final wealth of the median portfolio is higher than that of the EqW portfolio. If we look at table 1 and 2, then we can see some of these statistics in detail. The quartiles show that risk, e.g. the maximum portfolio loss, of the Median portfolio is lower than the risk of the CVaR portfolio, that is to say, lower than the portfolio that is considered to be the main benchmark for all our financial applications. As expected, the BestMed risk level is much lower than the BestMean. And if we look at table 2 and compare the averages of the HH and MAX indexes, we can see that the diversification obtained by the BestMed portfolio is even smaller than the CVaR model. This is quite remarkable, as the BestMed model is formulated without any risk measure whatsoever. Nevertheless, it still permits more diversification than a fully fledged model. If we look at the MAX index, we can also see that its value corresponds to a portfolio asset with maximum weight accounts for an average of 48% of the whole investment, showing that the optimal median is an *internal* point of the multidimensional simplex. This point is very far from the simplex corners, e.g. the best mean solution, and in the end, the optimal median model is achieved by processing different market information from the mean models.

The BestMed model alone is not enough to base investment decisions on as risk measures need to be introduced to the model. The data from the models MedMAD, MedVaR, MedCVaR and MedMAX is plotted in figures 6, 7, 8 and 9. As can be seen, all the models provide higher returns than the EqW portfolios, and also improve on the return performance of the BestMed model in three cases out of four, see table 1. Regarding risk, e.g. maximum loss risk, all the models improve on the BestMed model as they all achieve less risk. These higher returns can be explained by the fact that the median, when measuring allocation efficiency, discards both positive and *negative* jumps. The former may be seen as a positive feature, but the latter is negative, as the drops registered warn of more severe, future losses. However, these potentially bad investments are prevented by the risk measures. If we look

[8]This experiment also suggests that the good results of the MeanCVaR model with the same data are due to the CVaR constraints, e.g. the diversification imposed by the risk measure, and not due to the efficiency criteria, e.g. the Best Mean.

at the portfolio diversification indexes, see table 2, we can see that the four Median/Risk models enjoyed greater diversification than the BestMed and the minimum values, apart from the EqW, were obtained by the MedCVaR and MedMAD portfolios. The high diversification of the MedMAD portfolio will be further confirmed in the following tests too.

In terms of an overall conclusion to this first test, our data shows that Best Median portfolios are an important decision making tool as they achieved the highest returns on the EqW portfolio. Our data also shows that using the BestMean as the portfolio efficiency measure may lead to major drops, a fact of life that, if not carefully monitored, may impoverish the wealth of the portfolio. These drops can be avoided using any Median/Risk model because the resulting portfolio is more diversified, even than the ones that match the Mean with a risk measure, like the MeanCVaR. Having said that, the test also demonstrates that as far as risk is concerned, the EqW portfolio is still the best[9].

4.2 Beasley and Cesarone's time series

We then applied the same strategy outlined in the previous section to the public data sets proposed by Beasley *et al.* [6, 14] and Cesarone *et al.* [15]. This involves 13 data sets, covering different stock markets and time periods. The purpose of the test was not to show that there is a portfolio model that outperforms all the others in every market and every time period, as this would be an impossible task. What we really wanted to see was whether the main portfolio features would remain the same when we applied the fixed parameters used in the Milan experiments.

It is important to note that the 13 data sets are composed of weekly data instead of daily data, as was the case with the Milan experiment, therefore keeping the same estimation time window, e.g. $T = 21$, and rolling window of $W = 20$, may result in a suboptimal policy. But we prefered to maintain the same parameters as the Milan experiments, so the models could be compared on a completely neutral base. After all, if a model works well on a blind test, like this, then it can only improve when optimized on more realistic parameters.

Table 3, summarizes the data sources. As can be seen, they come from a wide timespan covering radically different economic climates, scenarios and markets. After the experiments, the most striking difference between the

[9]Note that the diversification provided by the optimization models uses market information only, but can be decreased further by imposing bounds on the maximum investment. This, however, would be an arbitrary new assumption for our experiments and was, therefore, not tested.

markets, is how effective the BestMean strategy is with regard to the following factors:

- Markets where investing is easy. In this case, simply investing all available money on a single asset is rewarded with the highest final wealth and is the most profitable strategy of all. This occurs on Beasley's 1, 2, 3, 4, 6 and 8 time series. The EqW strategy is the least risky, but it achieves lower returns. All the Median models are located somewhere in the middle.

- Time series where investing is difficult, and the BestMean is more risky than the EqW, and also achieves lower returns. These refer to Beasley's 5 and 7 time series, and all the Cesarone files (except the Nasdaq, see below). In all these cases, the Median models achieve higher returns than the EqW, but at the price of higher risk.

- Time series where investing is very difficult, e.g. the Cesarone Nasdaq time series, where no active strategies beat the EqW.

Diversification indexes: The two concentration indexes for each individual single test are shown in tables 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29. The most important result is that this data confirms the Milan results, showing that the median models all increase diversification. The median portfolio with no risk at all, still obtains half the concentration of the BestMean model. When risk measures are introduced, diversification improves, and we can see that the most diversified model is the MedMAD model, in which the MAX index range stretches approximately from 20 to 35%. Clearly, all these numbers are still a long way from the EqW diversification index.

The model with the greatest return: In table 30, column 1 shows which strategy guaranteed the highest profit for each data set. As can be seen, for 7 of the 13 data sets, BestMean was a rewarding choice, naturally, at the price of high risk, see below. But in 5 of the data sets, median portfolios perform better than both the BestMean and the EqW, and these are the cases in which the BestMean is worse than the EqW. In adverse scenarios, it therefore seems that median portfolios still outperform the EqW. The only exception is the last case, the Nasdaq, in which no portfolio is better than the EqW[10].

The least risky model: The second column in the table 30 shows which portfolios offer less risk. Here, the EqW is the most diversified portfolio, but

[10]Perhaps in this case our time windows were completely wrong, as the Nasdaq is often considered a very fast and reactive market.

ex-post is not always the least risky. It is the best in 9 cases out of 13, but in 4 cases, MedMAD diversification is more efficient. This shows that the diversification in Median models, even if it is higher than the EqW portfolio when measured through the diversification indexes, can still select assets with the lowest market risk.

Combining risk/return analysis: We have tried to summarize the quantile results in tables 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28 using risk/return analysis. The general principle of portfolio theory is that extra-profit is achieved at the price of higher risk, so if a model is not over-dominated by either risk or return measures, then it is a useful tool for decision making. First of all, in table 31, we can see that the return on the Median models is nearly always better than the EqW:

1. BestMed achieved a higher return than the EqW 8 times out of 13.
2. Median/Risk portfolios achieved a higher return in 43 matches out of 52.

Regarding point 1, the BestMean achieved a higher return than the EqW 7 times out of 13, so the BestMed only has a slight advantage. But this performance can be greatly improved by adding a risk measure, see point 2, as when this is done, in the vast majority of cases the Median/Risk models outperform the EqW.

When the Median models are compared with the BestMean, which is the most profitable strategy in 7 cases out of 13, we would expect the median models to have a much lower level of risk, as they are much more diversified. From the tests, see 32, we can see that:

1. BestMed has less risk than BestMean 8 times out of 13.
2. Median/Risk portfolios are less risky in 44 matches out of 52.

Regarding point 1, when the median is implemented without a risk measure, the results can be disappointing, as it registers more risk in 5 cases out of 13, even though its diversification is better. However, this result can be greatly improved when risk measures are included in the model: all risk comparisons between the Median/Risk models and the BestMean demonstrate that in the vast majority of cases the Median models achieve less risk.

One final remark must be devoted to the MedMAD strategy. We have seen that the diversification indexes are the lowest of all the active portfolios, and that in 2 experiments, see table 30, this is the dominating strategy, i.e. it achieves less risk and more returns than all the other portfolio models. If this data is then combined with the data in tables 31 and 32, it would seem fair to conclude that the MedMAD model is the best.

5 Conclusion

In this paper we have tested a number of portfolio models that use the Median instead of the Mean as the objective function. Theoretically, these portfolio models should be better when data is non-normal, and also, as the objective function has many local optima, they should provide more diversification. When the real data is tested, the results show that this is true. Median models do provide more portfolio diversification, and they can achieve higher profits. As we have already explained, this application has been specifically designed for cases is which only a limited amount of data is available for optimization, e.g. when T is much less than K, and Mean/Variance models cannot be used on account of the singularity of the covariance matrix. This is why we believe that the Median/MAD model is a valid reformulation of the Markovitz principle.

From these conclusions, two questions naturally arise. Firstly, what happens when the T is large? In this case, Median models can be directly compared to Mean/Variance models, but first we must extend the heuristic methods developed in [7, 26] to deal with the constrained optimal median problems involved too. And secondly, what happens if we use a robust risk estimator? We have constructed our models using the most popular linear risk function available in current literature, but only VaR can be interpreted as a robust measure, as it is an order statistic. A more interesting robust measure is the Median Absolute Deviation, but, as this again is a median model, it would increase the binary variables and consequently the complexity of the problem. Nevertheless, we are convinced that this is a line of research that merits future attention.

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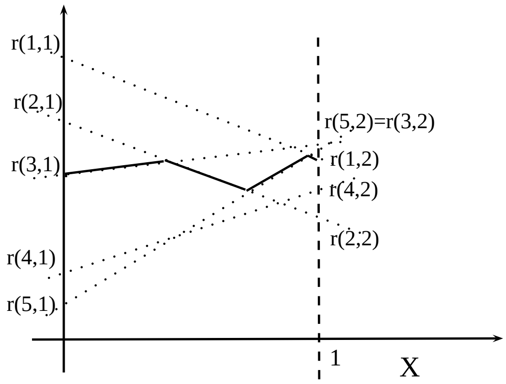

Figure 1: A graph showing the median of the convex combination of two assets.

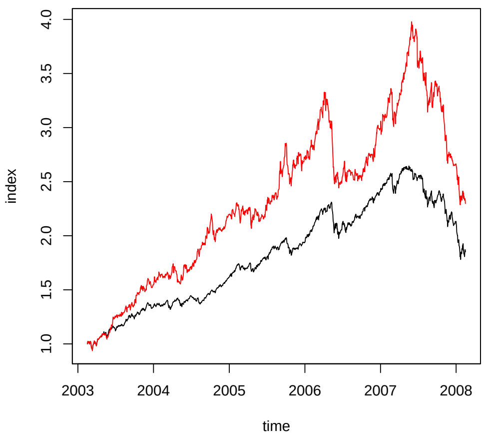

Figure 2: Milan stock market: comparing the EqW (black line) and MeanC-VaR (red line) portfolios.

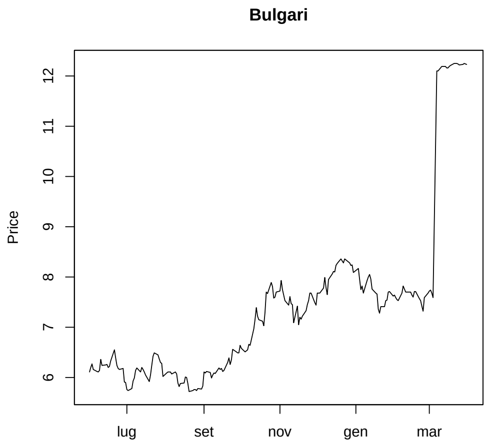

Figure 3: Bulgari prices before and after OPA.

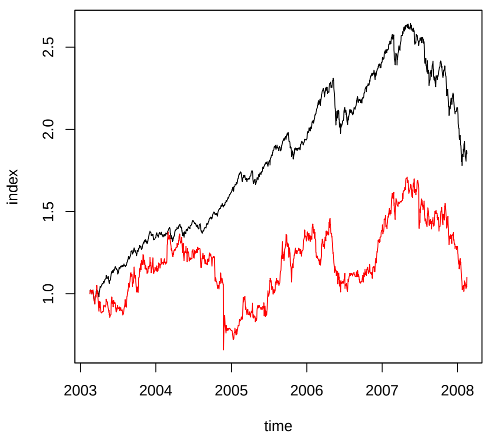

Figure 4: Milan stock market: comparing the EqW (black line) and the BestMean (red line) portfolios.

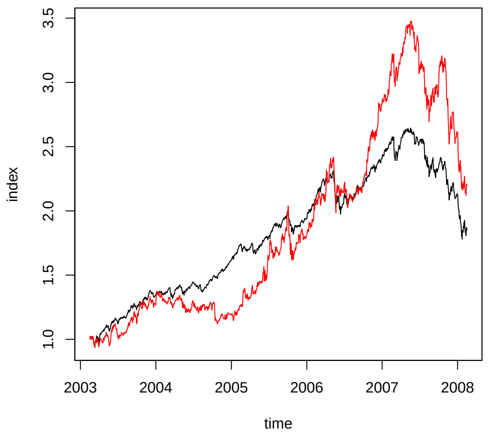

Figure 5: Milan stock market: comparing the EqW (black line) and BestMed (red line) portfolios.

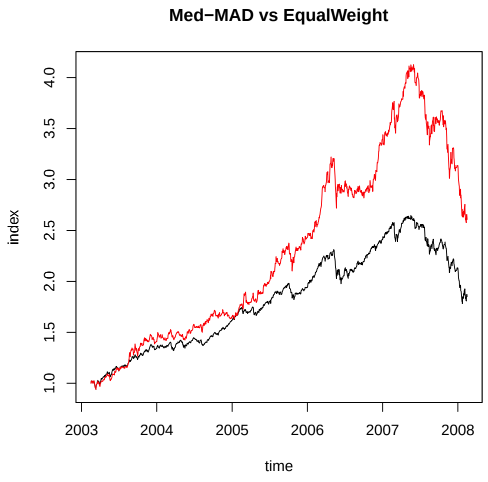

Figure 6: Milan stock market: comparing the EqW (black line) and the MedMAD (red line) portfolios.

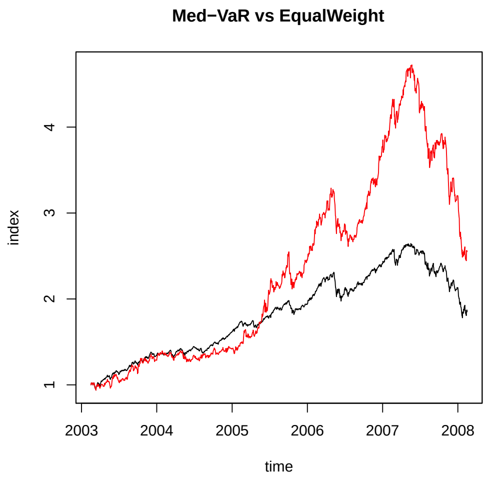

Figure 7: Milan stock market: comparing the EqW (black line) and the MedVaR (red line) portfolios

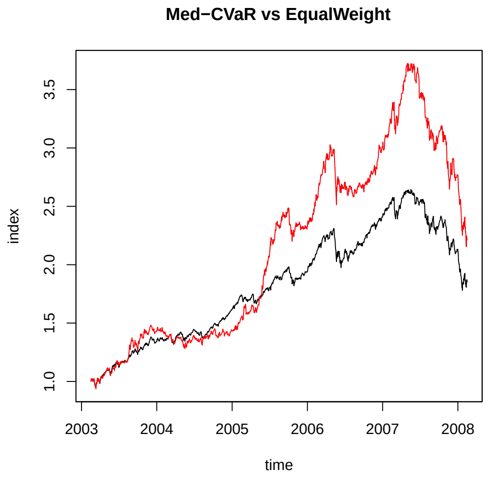

Figure 8: Milan stock market: comparing the EqW (black line) and the MedCVaR (red line) portfolios

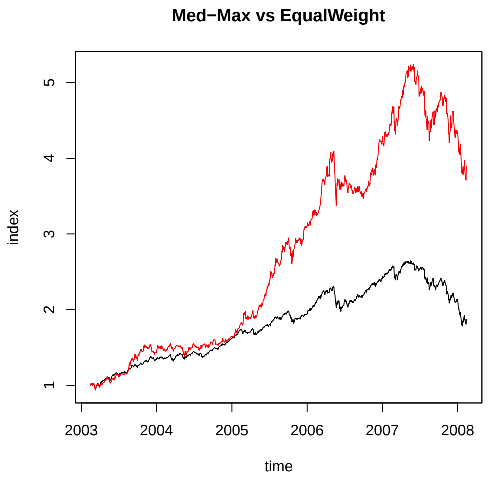

Figure 9: Milan stock market: comparing the EqW (black line) and the MedMAX (red line) portfolios

	Min.	1st Qu.	Median	Mean	3rd Qu.	Max.
Equal W	-0.0468200	-0.0029100	0.0010640	0.0005039	0.0049610	0.0305100
CVaR.50	-0.0688400	-0.0060630	0.0007623	0.0007139	0.0078750	0.0450600
Best Med	-0.0580500	-0.0071540	0.0004696	0.0007094	0.0090390	0.0639800
Best Mean	-0.377500	-0.010730	0.000000	0.000353	0.010890	0.118800
MedVar	-0.0574100	-0.0063020	0.0007768	0.0008085	0.0084260	0.0647400
MedCVaR	-0.0565500	-0.0054030	0.0007629	0.0006752	0.0069260	0.0582200
MedMax	-0.0554400	-0.0051970	0.0009366	0.0011130	0.0081340	0.0624700
MedMAD	-0.0516400	-0.0047880	0.0006076	0.0007968	0.0068860	0.0533700

Table 1: Milan stock data: Mean and quantile returns from different portfolio models

	HH	max
Equal W	0.01666667	0.01666667
CVaR.50	0.3912677	0.5012798
Best Med	0.3404817	0.4834777
Best Mean	1	1
MedVar	0.302726	0.4454241
MedCVaR	0.2255899	0.3518155
MedMax	0.2566656	0.3837267
MedMAD	0.2293793	0.3526348

Table 2: Milan stock data: Mean and quantile returns from different portfolio models

Source	Market	T, K	Period
Beasley-1	Hang Seng	290, 31	March 1992, September 1997
Beasley-2	Dax 100	290, 85	March 1992, September 1997
Beasley-3	FTSE 100	290, 89	March 1992, September 1997
Beasley-4	S&P 100	290, 98	March 1992, September 1997
Beasley-5	Nikkey 225	290, 225	March 1992, September 1997
Beasley-6	S&P 500	290, 457	March 1992, September 1997
Beasley-7	Russell 2000	290, 1318	March 1992, September 1997
Beasley-8	Russell 3000	249, 2151	March 1992, September 1997
Cesarone-1	EuroStoxx50	264,47	March 2003, March 2008
Cesarone-2	FTSE 100	264, 76	March 2003, March 2008
Cesarone-3	MIBTEL	264, 221	March 2003, March 2008
Cesarone-4	S&P 476	264, 476	March 2003, March 2008
Cesarone-5	NASDAQ	264, 2191	March 2003, March 2008

Table 3: Data sets from Beasley and Cesarone.

	Min.	1st Qu.	Median	Mean	3rd Qu.	Max.
EqW	-0.137200	-0.015530	0.005178	0.003510	0.022910	0.117500
BestMean	-0.147500	-0.026570	0.009221	0.011610	0.045000	0.219900
BestMed	-0.150000	-0.021820	0.009354	0.008079	0.035440	0.203600
MedVaR	-0.190700	-0.022160	0.008490	0.008261	0.036960	0.173100
MedMax	-0.149100	-0.020380	0.006472	0.003675	0.029080	0.116700
MedCVaR	-0.161600	-0.017160	0.005322	0.004303	0.026650	0.108100
MedMAD	-0.132800	-0.017800	0.006497	0.003748	0.028000	0.115600

Table 4: Mean and quantiles of portfolio strategies applied to the Hang Seng 1992-97

	HH	MAX
Equal W	0.03125	0.03125
Best Med	0.4624018	0.5947217
Best Mean	1	1
MedVaR	0.4295522	0.5492874
MedMax	0.2840001	0.419751
MedCVaR	0.2516018	0.3667023
MedMAD	0.2580075	0.3873757

Table 5: Diversification indexes of portfolio strategies applied to the Hang Seng 1992-97.

	Min.	1st Qu.	Median	Mean	3rd Qu.	Max.
EqW	-0.051250	-0.007485	0.001643	0.001501	0.011430	0.049150
BestMean	-0.149500	-0.013980	0.002460	0.009176	0.034810	0.262400
BestMed	-0.090710	-0.008500	0.0007659	0.005814	0.023470	0.095680
MedVaR	-0.099840	-0.008776	0.002301	0.006348	0.021220	0.098920
MedMax	-0.093140	-0.008303	0.003446	0.005619	0.019920	0.095680
MedCVaR	-0.079940	-0.008924	0.001577	0.005536	0.020990	0.095680
MedMAD	-0.076020	-0.008675	0.002654	0.004210	0.017500	0.074930

Table 6: Mean and quantiles of portfolio strategies applied to the Dax 100, 1992-97

	HH	MAX
Equal W	0.01162791	0.01162791
Best Med	0.3900327	0.5325823
Best Mean	1	1
MedVaR	0.3263508	0.47111374
MedMax	0.3306594	0.4744355
MedCVaR	0.2804148	0.4182394
MedMAD	0.1744027	0.2616575

Table 7: Diversification indexes of portfolio strategies applied to the Dax 100, 1992-97

	Min.	1st Qu.	Median	Mean	3rd Qu.	Max.
EqW	-0.054780	-0.008188	0.002756	0.002658	0.012680	0.086870
BestMean	-0.111600	-0.018270	0.003419	0.006091	0.028760	0.159700
BestMed	-0.069700	-0.013230	0.002998	0.003396	0.018810	0.101300
MedVaR	-0.081350	-0.012800	0.003391	0.003819	0.017030	0.101300
MedMax	-0.061130	-0.010270	0.003145	0.004056	0.017200	0.094070
MedCVaR	-0.060520	-0.010780	0.002083	0.003422	0.014590	0.100200
MedMAD	-0.054220	-0.009937	0.002370	0.003726	0.015620	0.086870

Table 8: Mean and quantiles of portfolio strategies applied to the FTSE 100, 1992-97.

	HH	MAX
Equal W	0.01111111	0.01111111
Best Med	0.3467349	0.5033554
Best Mean	1	1
MedVaR	0.2967019	0.4552982
MedMax	0.2198074	0.3434669
MedCVaR	0.2220292	0.3501135
MedMAD	0.1711999	0.2749024

Table 9: Diversification indexes of portfolio strategies applied to the FTSE 100, 1992-97.

	Min.	1st Qu.	Median	Mean	3rd Qu.	Max.
EqW	-0.036100	-0.007154	0.003276	0.002874	0.012450	0.053120
BestMean	-0.168800	-0.024930	0.002924	0.005857	0.036850	0.166000
BestMed	-0.09510	-0.02038	-0.00007669	0.003054	0.02283	0.1597
MedVaR	-0.116800	-0.018910	0.002750	0.003435	0.024480	0.148200
MedMax	-0.082720	-0.013790	0.002103	0.002206	0.017990	0.067900
MedCVaR	-0.094520	-0.013270	0.002513	0.003132	0.021440	0.096160
MedMAD	-0.101600	-0.010510	0.003163	0.002233	0.016110	0.076040

Table 10: Mean and quantiles of portfolio strategies applied to the S&P 100, 1992-97

	HH	MAX
Equal W	0.01010101	0.01010101
Best Med	0.4055683	0.5033554
Best Mean	1	1
MedVaR	0.3277657	0.4729854
MedMax	0.1894146	0.3164776
MedCVaR	0.1811819	0.2986034
MedMAD	0.1768587	0.2960193

Table 11: Diversification indexes of portfolio strategies applied to the S&P 100, 1992-97

	Min.	1st Qu.	Median	Mean	3rd Qu.	Max.
EqW	-0.12600	-0.02008	-0.000766	-0.0015030	0.0167600	0.1215000
BestMean	-0.135200	-0.031700	-0.005603	-0.001971	0.023160	0.141700
BestMed	-0.238400	-0.025160	-0.003819	-0.001535	0.022340	0.145500
MedVaR	-0.121900	-0.022090	-0.002978	0.0001875	0.019580	0.14750
MedMax	-0.127700	-0.022660	-0.003733	-0.0004099	0.019540	0.12840
MedCVaR	-0.105400	-0.022110	-0.003827	-0.0004266	0.018150	0.13230
MedMAD	-0.084180	-0.019890	-0.002876	-0.0003084	0.018040	0.11810

Table 12: Mean and quantiles of portfolio strategies applied to the Nikkey 225, 1992-97

	HH	max
Equal W	0.004424779	0.004424779
Best Med	0.353322	0.4848529
Best Mean	1	1
MedVaR	0.3041650	0.4267809
MedMax	0.2726755	0.3958407
MedCVaR	0.2550025	0.3825
MedMAD	0.2149165	0.3294858

Table 13: Diversification indexes of portfolio strategies applied to the Nikkey 225, 1992-97

	Min.	1st Qu.	Median	Mean	3rd Qu.	Max.
EqW	-0.097090	-0.012720	0.004244	0.001724	0.016430	0.079340
BestMean	-0.262500	-0.041820	0.008067	0.010360	0.064700	0.234800
BestMed	-0.209100	-0.040520	0.010290	0.008092	0.056830	0.212500
MedVaR	-0.242100	-0.034370	0.004177	0.006049	0.054520	0.212500
MedMax	-0.251700	-0.019470	0.006566	0.005517	0.039300	0.160600
MedCVaR	-0.241600	-0.018830	0.005330	0.005735	0.035360	0.179900
MedMAD	-0.199400	-0.017070	0.004812	0.004704	0.028680	0.121800

Table 14: Mean and quantiles of portfolio strategies applied to the S&P 500, 1992-97

	HH	MAX
Equal W	0.002183406	0.002183406
Best Med	0.4109845	0.553372
Best Mean	1	1
MedVaR	0.3717943	0.5068179
MedMax	0.2137783	0.3372625
MedCVaR	0.1897302	0.3158391
MedMAD	0.1634776	0.2586778

Table 15: Diversification indexes of portfolio strategies applied to the S&P 500, 1992-97

	Min.	1st Qu.	Median	Mean	3rd Qu.	Max.
EqW	-0.126000	-0.020080	-0.0007666	-0.0015030	0.0167600	0.1215000
BestMean	-0.135200	-0.031700	-0.005603	-0.001971	0.023160	0.141700
BestMed	-0.238400	-0.025160	-0.003819	-0.001535	0.022340	0.145500
MedVaR	-0.121900	-0.022090	-0.002978	0.0001875	0.019580	0.147500
MedMax	-0.127700	-0.022660	-0.003733	-0.0004099	0.019540	0.128400
MedCVaR	-0.105400	-0.022110	-0.003827	-0.0004266	0.018150	0.132300
MedMAD	-0.084180	-0.019890	-0.002876	-0.0003084	0.018040	0.118100

Table 16: Mean and quantiles of portfolio strategies applied to the Russell 200, 1992-97

	HH	MAX
Equal W	0.004424779	0.004424779
Best Med	0.353322	0.0.4848529
Best Mean	1	1
MedVaR	0.3041650	0.4267809
MedMax	0.2726755	0.3958407
MedCVaR	0.2550025	0.3825
MedMAD	0.2149165	0.3294858

Table 17: Diversification indexes of portfolio strategies applied to the Russell 200, 1992-97

	Min.	1st Qu.	Median	Mean	3rd Qu.	Max.
EqW	-0.099920	-0.011120	0.004791	0.001604	0.015450	0.094400
BestMean	-0.37040	-0.05557	0.01185	0.01243	0.07950	0.46570
BestMed	-0.429400	-0.029020	0.008644	0.003490	0.044740	0.341000
MedVaR	-0.275900	-0.033380	0.010980	0.008729	0.054710	0.320700
MedMax	-0.438400	-0.027920	0.008186	0.003596	0.045550	0.214100
MedCVaR	-0.412900	-0.025400	0.005969	0.002960	0.043750	0.226900
MedMAD	-0.216500	-0.017850	0.012490	0.006061	0.039950	0.154500

Table 18: Mean and quantiles of portfolio strategies applied to the Russell 300, 1992-97

	HH	MAX
Equal W	0.004424779	0.004424779
Best Med	0.2656127	0.3964605
Best Mean	1	1
MedVaR	0.2239116	0.3682222
MedMax	0.1892909	0.3336972
MedCVaR	0.1854615	0.3250739
MedMAD	0.1345886	0.2310935

Table 19: Diversification indexes of portfolio strategies applied to the Russell 300, 1992-97

	Min.	1st Qu.	Median	Mean	3rd Qu.	Max.
EqW	-0.060490	-0.009208	0.004439	0.003216	0.017340	0.075660
BestMean	-0.712100	-0.01457	0.003656	0.002990	0.022710	0.762400
BestMed	-0.101800	-0.012080	0.003844	0.002888	0.018770	0.077920
MedVaR	-0.100600	-0.010880	0.003594	0.002901	0.019080	0.083540
MedMax	-0.084970	-0.012560	0.003976	0.003182	0.019460	0.075660
MedCVaR	-0.082050	-0.010090	0.004851	0.004872	0.017880	0.446500
MedMAD	-0.079710	-0.009283	0.003161	0.003331	0.017750	0.075660

Table 20: Mean and quantiles of portfolio strategies applied to the EuroStoxx, 2003-08

	HH	MAX
Equal W	0.020833	0.020833
Best Med	0.3133755	0.4384548
Best Mean	1	1
MedVaR	0.2615966	0.3870369
MedMax	0.2663146	0.3871546
MedCVaR	0.2438328	0.3655124
MedMAD	0.201106	0.3047439

Table 21: Diversification indexes of portfolio strategies applied to the EuroStoxx, 2003-08

	Min.	1st Qu.	Median	Mean	3rd Qu.	Max.
EqW	-0.065450	-0.006498	0.004983	0.003182	0.016040	0.068650
BestMean	-0.302600	-0.022780	0.007054	0.003655	0.033300	0.153500
BestMed	-0.218700	-0.017580	0.005245	0.008551	0.028930	1.130000
MedVaR	-0.181000	-0.018390	0.007401	0.008166	0.029520	0.980400
MedMax	-0.108800	-0.014170	0.005228	0.003752	0.025100	0.120000
MedCVaR	-0.104400	-0.011000	0.005521	0.004423	0.022960	0.111700
MedMAD	-0.093290	-0.007530	0.007398	0.004961	0.020740	0.096240

Table 22: Mean and quantiles of portfolio strategies applied to the FTSE 100, 2003-08

	HH	max
Equal W	0.012658	0.012658
Best Med	0.3842628	0.518389
Best Mean	1	1
MedVaR	0.3385556	0.483172
MedMax	0.2520822	0.3891488
MedCVaR	0.2329686	0.3628687
MedMAD	0.1934227	0.3169802

Table 23: Diversification indexes of portfolio strategies applied to the FTSE 100, 2003-08

	Min.	1st Qu.	Median	Mean	3rd Qu.	Max.
EqW	-0.078020	-0.006822	0.004380	0.001378	0.01264	0.050430
BestMean	-0.36080	-0.02327	0.00000	0.00096	0.01771	0.45320
BestMed	-0.801100	-0.020960	0.0005164	-0.0033710	0.02255	0.3732000
MedVaR	-0.537000	-0.018740	0.003674	0.0005118	0.02482	0.3187000
MedMax	-0.393500	-0.013830	0.004793	0.002709	0.02494	0.339200
MedCVaR	-0.385100	-0.013860	0.004835	0.002391	0.02213	0.330100
MedMAD	-0.193800	-0.010050	0.006839	0.003839	0.01968	0.216200

Table 24: Mean and quantiles of portfolio strategies applied to the MIBTEL, 2003-08

	HH	MAX
Equal W	0.0044247	0.0044247
Best Med	0.3415395	0.4775823
Best Mean	1	1
MedVaR	0.2908776	0.4376798
MedMax	0.2040393	0.328877
MedCVaR	0.1942376	0.3236113
MedMAD	0.1387325	0.2243378

Table 25: Diversification indexes of portfolio strategies applied to the MIBTEL, 2003-08

	Min.	1st Qu.	Median	Mean	3rd Qu.	Max.
EqW	-0.061440	-0.008898	0.003767	0.002511	0.014900	0.072890
BestMean	-0.228000	-0.033560	0.010220	0.002763	0.049020	0.190100
BestMed	-0.157300	-0.017660	0.006283	0.005080	0.036560	0.150500
MedVaR	-0.148700	-0.017660	0.008731	0.006318	0.037330	0.134400
MedMax	-0.136600	-0.015090	0.005941	0.004086	0.025700	0.118000
MedCVaR	-0.155200	-0.013020	0.006160	0.003786	0.027840	0.120600
MedMAD	-0.109200	-0.009447	0.006418	0.004077	0.023450	0.077700

Table 26: Mean and quantiles of portfolio strategies applied to the S&P 500, 2003-08

	HH	MAX
Equal W	0.0021	0.0021
Best Med	0.3265198	0.4634541
Best Mean	1	1
MedVaR	0.3076458	0.4325518
MedMax	0.2451123	0.3704486
MedCVaR	0.2629144	0.3889473
MedMAD	0.1710927	0.2840422

Table 27: Diversification indexes of portfolio strategies applied to the S&P 500, 2003-08

	Min.	1st Qu.	Median	Mean	3rd Qu.	Max.
EqW	-0.062920	-0.009319	0.001967	0.001792	0.015270	0.056850
BestMean	-0.628600	-0.058370	-0.001080	-0.005323	0.046610	0.916300
BestMed	-0.230800	-0.044730	0.002729	-0.0005791	0.035370	0.257400
MedVaR	-0.238200	-0.032390	-0.000899	-0.001441	0.0306000	0.352300
MedMax	-0.228600	-0.033320	-0.001126	-0.001285	0.030960	0.185400
MedCVaR	-0.203000	-0.028240	0.002465	0.001013	0.035070	0.152800
MedMAD	-0.154100	-0.026230	0.005575	0.001253	0.033020	0.118000

Table 28: Mean and quantiles of portfolio strategies applied to the NASDAQ, 2003-08

	HH	MAX
Equal W	0.000455	0.000455
Best Med	0.2946113	0.4369417
Best Mean	1	1
MedVaR	0.2431236	0.3992313
MedMax	0.1778284	0.2920459
MedCVaR	0.1744571	0.3016559
MedMAD	0.1187219	0.204928

Table 29: Diversification indexes of portfolio strategies applied to the NASDAQ, 2003-08

Data	Best return	Least Risk
Beasley-1	BestMean	MedMAD
Beasley-2	BestMean	EqW
Beasley-3	BestMean	MedMAD
Beasley-4	BestMean	EqW
Beasley-5	MedMAD	MedMAD
Beasley-6	BestMean	EqW
Beasley-7	MedMAD	MedMAD
Beasley-8	BestMean	EqW
Cesarone-1	MedCVaR	EqW
Cesarone-2	BestMed	EqW
Cesarone-3	MedMAD	EqW
Cesarone-4	MedVAR	EqW
Cesarone-5	EqW	EqW

Table 30: The highest return portfolios and least risk portfolios.

Data	Most return	Less return
Beasley-1	All	None
Beasley-2	All	None
Beasley-3	All	None
Beasley-4	MedVaR, MedCVaR	MedMax, MedMAD
Beasley-5	All	None
Beasley-6	All	None
Beasley-7	All	None
Beasley-8	All	None
Cesarone-1	MedCVaR, MedMAD	MedVaR, MedMax
Cesarone-2	All	None
Cesarone-3	MedMAD, MedMax, MedCVaR	MedVaR
Cesarone-4	All	None
Cesarone-5	None	All

Table 31: Return comparison of all Median/Risk portfolio models against EqW

Data	Less Risk	More Risk
indtrack-1	MedMAD	MedVaR, MedMax, MedCVaR
indtrack-2	All	None
indtrack-3	All	None
indtrack-4	All	None
indtrack-5	All	None
indtrack-6	All	None
indtrack-7	All	None
indtrack-8	MedMAD, MedVaR	MedMax, MedCVaR
EUROSTOXX	All	None
FTSE	All	None
MIBTEL	MedMAD	MedVaR, MedMax, MedCVaR
S&P	All	None
NASDAQ	All	None

Table 32: Risk comparison of all Median/Risk portfolio models against Best-Mean