On the effect of ideology in proportional representation systems

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Abstract

In this paper we propose a model in which there are ideological and strategic voters who vote under poportional rule. We prove that the behavior of ideological voters matters, in that it affects the outcome. We also show how a subset of strategic voters changes his voting behavior to balance the ideological players' votes. However, they can only partially adjust. Strategic voters will vote accordingly to this cutpoint outcome: any strategic voter on its right votes for the rightmost party and any strategic voter on its left votes for the leftmost party.

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1 Introduction

In this paper we present a model that is an extension of De Sinopoli and Iannantuoni (2007). In the former the authors study strategic voting under proportional rule. The main result shows that essentially only a two-party equilibrium exists, in which rational voters vote only for the two extremist parties. In this paper we provide an analysis in case there are some voters who simply cast their ballot in favor of their preferred party. We define those voters as ideological ones. We believe that this question is interesting in various respect. Firstly, most of the voting literature (Shepsle 1991, Cox 1997, Persson and Tabellini 2000) deals with models in which either all voters are strategic, either all of them are ideological. Hence, it seems interesting to study the case in which both types of voters coexist. Secondly, the assumption that each voter is rational might be

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considered too strong. Hence, by assuming that there are strategic and ideological voters, we believe to provide a more "realistic" model. Finally, it would be possible to better reconcile the results of De Sinopoli and Iannantuoni (2007), i.e. a two-party equilibrium, with the general agreement on the evidence that proportional representation election implies multipartyism (see Cox 1997)[1].

Specifically, we study a society composed by policy motivated strategic citizens and by ideological citizens who vote for one of a finite number of parties by proportional rule. Given the electoral result, we define the policy outome as the linear combination of parties' position weighted by the share of votes each party gets at the election. In this context, we want to study the strategic voting behavior in presence of the ideological voters' behavior. We address three main questions. The first one is if, and how, the ideological voters' behavior affects the outcome. The second one is how strategic voters will "respond" to that. The third question concerns the number of parties arising at equilibrium.

We prove that the behavior of ideological voters matters, in that it affects the outcome. We prove that that the policy will, in general, be different with respect to the case when all voters act strategically. This result holds even in presence of a arbitrarily small number of ideological voters. Concerning the second question, we show how a subset of strategic voters changes his voting behavior to balance the ideological players' votes. However, they can only partially adjust. Strategic voters will vote in accord with such a cutpoint outcome: in equilibrium any strategic voter on its right votes for the rightmost party and any strategic voter on its left votes for the leftmost party. The intuition of the result is the following. Given the ideological voting behavior, strategic voters misrepresent their preferences by voting for the extremist parties in order to drag the policy outcome toward their bliss policy. Finally, obviously we cannot conclude that only two parties emerge at equilibrium, being the number of parties depending also from the ideological voters' behavior.

The paper is organized as follow. In section 2 we describe the model; we analyze the pure strategy equilibria, and, then, the mixed strategy ones in section 3; we present an example in section 4, and section 5 concludes.

2 The Model

Policy space. The policy space $\mathbb{X}$ is a closed interval of the real line and, without lost of generality, we assume $\mathbb{X} = [0, 1]$.

Parties. Let us assume that there is an exogenously given set of parties $M = \{1, ..., k, ...m\}$ $(m \geq 2)$, indexed by k. Each party k is characterized by a policy $\zeta_k \in [0, 1]$.

Voters. There is a finite set of voters $N = \{1, ..., i, ...n\}$. Each voter i, characterized by a bliss point $\theta_i \in \Theta = [0, 1]$, has single peaked preferences. The set of voters N is partitioned in two subsets N^ρ and N^ι, denoting respectively

[1] Even if there are striking counterexamples to this general agreement, see Cox 1997.

the set of strategic and ideological voters. We indicate the cardinality of N^ρ by n^ρ, and the cardinality of N^ι by n^ι. Hence, $n = n^\rho + n^\iota$.

Strategic voters. First let us stress that the single peakedness of voters' preferences is the only assumption needed to reach the result for pure strategy equilibrium. To analyze also mixed strategy equilibria, we assume that it exists a fundamental utility function (à la Harsany) $u : \Re^2 \to \Re$, continuously differentiable with respect to the first argument[2], which represents the preferences, that is $u_i(X) = u(X, \theta_i)$. Since each voters can cast his vote for any party, the pure strategy space of each player $i \in N^\rho$ is $S_i = \{1, ..., k, ..., m\}$ where each $k \in S_i$ is a vector of m components with all zeros except for a one in position k, which represents the vote for party k. A mixed strategy of player i is a vector $\sigma_i = (\sigma_i^1, ...\sigma_i^k, ..., \sigma_i^m)$ where each σ_i^k represents the probability that player i votes for party k.

Ideological voters. A natural way to model ideological voters it is to assume that their strategy space is degenerate, coinciding with the vote in favor of their preferred party We denote by N_k^ι the set of ideological voters who vote for party k and with n_k^ι its cardinality. Hence, $s_i = k \; \forall i \in N_k^\iota$, and $n^\iota = \sum\limits_{k=1}^{m} n_k^\iota$.

Proportional rule and the policy outcome. Given a pure strategy combination[3] $s = (s_1, s_2, ..., s_n)$, we define $v(s) = \sum\limits_{i \in N} \frac{s_i}{n}$ as the vector representing for each party its share of votes. We capture the spirit of proportional representation by assuming that any party in parliament participates to the policy determination with a strenght equal to its share of votes. The policy outcome is a linear combination of parties' policies each coefficient being equal to the corresponding share of votes.[4] Hence, the policy outcome can be written as:

$$X(s) = \sum_{k=1}^{m} \zeta_k v_k(s). \tag{1}$$

Given the set of parties and the utility function u, a finite game is characterized by the set of voters N, the subset of strategic voters and their bliss points, as well as the subsets of ideological voters:

$$\Gamma = \left\{ N, N^\rho, \{\theta_i\}_{i \in N^\rho}, \{N_k^\iota\}_{k \in M} \right\}$$

[2] Hence, by single-peakedness, $\forall \bar{x}_2 \in [0,1]$, $\frac{\partial u(x_1, \bar{x}_2)}{\partial x_1} \gtreqless 0$ for $x_1 \lesseqgtr \bar{x}_2$ and $x_1 \in [0,1]$.

[3] In this model we do not allow for abstention. We cannot claim that this assumption is neutral. In our proof we use the fact that, as the number of players goes to infinity, the weight of each player goes to zero, and this does not hold if a large number of voters abstains.

[4] The linear outcome function it is the utilitarian solution of a bargaining process among politicians with a quadratic loss function. Hence, it is the result of a bargaining process of government formation *á la* Baron and Diermeier (2001), under the assumption that the status quo is quite negative for parliamentary members. This is a weak assumption if the status quo is given by new election where parliamentary members face the risk of not being reelected, and the cost of staying out of the legislature is sufficiently large, as in Austen-Smith and Banks (1988).

Finally, we define $H^{\rho}(\theta)$ the distribution of the strategic voters' bliss points.

3 The equilibrium

Given the above defined game Γ and the strategic voters' bliss policies distribution $H^{\rho}(\theta)$, we analyze strategic voters' behavior when only pure strategies are allowed. The reason to first study only pure strategy equilibria is to better show the intuition behind the result: rational voters have an incentive to vote for the extremist parties in order to drag the policy outcome toward their favored policy. Specifically, in any pure strategy Nash equilibrium strategic voters vote only for the extreme parties, except for a neighborhood inversely related to the total number of players. We then define the cutpoint outcome, that is the outcome obtained with any strategic voter strictly on its right voting for the rightmost party and any strategic voter strictly on its left voting for the leftmost party. Such a strategy combination is a pure strategy Nash equilibrium of the game, if no player's bliss policy coincides with the cutpoint outome. Moreover, the cutpoint outcome so defined is in general different from the cutpoint outcome defined in De Sinopoli and Iannantuoni (2007), in that there is a "fixed effect" of the ideological voters' behavior for which strategic voters cannot fully adjust.

As nothing assures us that the sufficient condition above for the existence of a pure strategy equilibrium is satisfied, or that mixed strategy equilibria do not behave completely differently, we extend the analysis to the case when voters are allowed to play mixed strategies. The result implies that in any mixed strategy equilibrium, except for a neighborhood inversely related to the total number of players, strategic voters vote for the extremist parties. Moreover, we prove the main result of this paper: in any equilibrium any strategic player on the right of the cutpoint outcome votes for the rightmost party, and any strategic player on the left of the cutpoint outcome votes for the leftmost party, except for a neighborhood inversely related to the total number of voters.

In order to simplify the notation, in the following we will denote L the most leftist party and R the most rightist (i.e., $L = \arg\min_{k\in M} \zeta_k$, $R = \arg\max_{k\in M} \zeta_k$).[5]

3.1 Pure strategy equilibria

Let us start by analyzing the pure strategy equilibrium. First, we underline that only the assumption of single peakedness of strategic voters' preferences is needed to get the result. We prove that every pure strategy equilibrium is such that (except for a neighborhood whose length is inversely proportional to the total number of players) every strategic voter votes for one of the two extremist parties.

[5] We assume that there is only one party in ζ_L as well as in ζ_R. This assumption simplifies the notation, but it does not affect the result. Without this assumption, if we denote L and R the set of extremist parties, everything still holds.

Proposition 1 *Let s be a pure strategy equilibrium of a game Γ with n voters:*
(α) $\forall i \in N^\rho$, if $\theta_i \leq X(s) - \frac{1}{n}(\zeta_R - \zeta_L)$ then $s_i = L$,
(β) $\forall i \in N^\rho$, if $\theta_i \geq X(s) + \frac{1}{n}(\zeta_R - \zeta_L)$ then s_i=R.

Proof. [6](α) Notice that if $X(s_{-i}, L) \geq \theta_i$ then, by single-peakedness, L is the only best reply, for player i, to s_{-i} (i.e., $\forall k \neq L$, $X(s_{-i}, k) > X(s_{-i}, L)$). Since $X(s_{-i}, L) = X(s) - \frac{1}{n}(\zeta_{s_i} - \zeta_L) \geq X(s) - \frac{1}{n}(\zeta_R - \zeta_L)$, the assumption $\theta_i \leq X(s) - \frac{1}{n}(\zeta_R - \zeta_L)$, implies that L is the unique best reply, for player i, to s_{-i}. (β) A symmetric argument holds. ■

At this point, it is natural to analyze the strategy combination such that any strategic voter strictly on the left of the policy outcome votes for L, and any strategic voter strictly on the right of the policy outcome votes for R. We provide the following definition:

Definition 2 *Cutpoint policy outcome. Given a game Γ and the distribution of strategic voters' bliss points $H^\rho(\theta)$, let $\tilde{\theta}_\rho^\Gamma$, defined as cutpoint policy, be the unique policy outcome obtained with strategic voters strictly on its left voting for L and strategic voters strictly on its right voting for R, i.e. let $\tilde{\theta}_\rho^\Gamma$ be implicitly defined by:*

$$\tilde{\theta}_\rho^\Gamma \in \frac{n^\rho}{n}\left(\zeta_L \bar{H}^\rho\left(\tilde{\theta}_\rho^\Gamma\right) + \zeta_R\left(1 - \bar{H}^\rho\left(\tilde{\theta}_\rho^\Gamma\right)\right)\right) + \frac{n^\iota}{n}\sum_{k=1}^{m}\frac{n_k^\iota}{n^\iota}\zeta_k$$

where $\bar{H}^\rho$ is the correspondence defined by $\bar{H}^\rho(\theta) = \left[\lim_{y\to\theta^-}\bar{H}^\rho(y), \bar{H}^\rho(\theta)\right]$.

In the expression defining the cutpoint policy outcome is clearly depictable the roles of the ideological and rational voters on the policy. The first term of the right-hand side of the above expression represents the effect of the strategic voters' behavior, weighted by the share of the strategic voters on the cutpoint, while the second term is the "fixed" effect of the ideological voters' behavior, weighted by the share of ideological voters on the total number of voters.

Let us assume that no strategic player's preferred policy coincides with the cutpoint outcome. If all strategic players vote accordingly with the cutpoint, no strategic player on its left/right has an incentive to vote for any party different from L/R, because doing so would push the policy outcome further away from his preferred policy. We can, then, state the following proposition:

Proposition 3 *If $\theta_i \neq \tilde{\theta}_\rho^\Gamma$ $\forall i \in N^\rho$, then the strategy combination given by*
a) $s_i = L$ $\quad \forall i \in N^\rho$ with $\theta_i < \tilde{\theta}_\rho^\Gamma$
b) $s_i = R$ $\quad \forall i \in N^\rho$ with $\theta_i > \tilde{\theta}_\rho^\Gamma$
c) $s_i = k$ $\quad \forall i \in N_k^\iota$
is a pure strategy Nash equilibrium of the game Γ.

[6] This proof, as well as the others, goes in the same spirit of De Sinopoli and Iannantuoni (2007).

It is clear that nothing assures us that pure strategy equilibria exist; moreover we have to check if mixed strategy equilibria prescribe a different behavior for strategic voters.

3.2 Mixed strategy equilibria

We move now to the case when strategic voters are allowed to play mixed strategies. In order to undertake this analysis we have to assume also that the utility function u is continuously differentiable with respect to the first argument.

We recall that, given the set of candidates M and the utility function u, a game Γ is characterized by the set of players, the set of strategic voters and their bliss points, as well as the set of ideological voters. Let $\sigma = (\sigma_1, ..., \sigma_n)$ and $\bar{\mu}^\sigma = \sum_{i \in N} \frac{\sigma_i}{n}$. With abuse of notation, let $X(\bar{\mu}^\sigma) = \sum_{k=1}^{m} \zeta_k \bar{\mu}_k^\sigma$.

We can state the following proposition:

Proposition 4 *$\forall \varepsilon > 0$, $\exists n_0$ such that $\forall n \geq n_0$ if σ is a Nash equilibrium of a game Γ with n voters then:*
(α) $\forall i \in N^\rho$, if $\theta_i \leq X(\bar{\mu}^\sigma) - \varepsilon$ then $\sigma_i = L$
(β) $\forall i \in N^\rho$, if $\theta_j \geq X(\bar{\mu}^\sigma) + \varepsilon$ then $\sigma_j = R$.

Proof. See Appendix. ■

In the appendix we will show that $\bar{\mu}^\sigma$ is the expected share of votes. The proposition above says that in any Nash equilibrium, except for a neighborhood whose length decreases as the total number of players increases, every strategic voter to the left of $X(\bar{\mu}^\sigma)$ votes for L, while every strategic voter to the right votes for R.

Using the definition of cutpoint policy outcome, we can state the main result of this paper, that is "basically" an unique Nash equilibrium of the game exists:

Corollary 5 *$\forall \eta > 0$, $\exists n_1$ such that $\forall n \geq n_1$ if σ is a Nash equilibrium of a game Γ with n voters then:*
(α) $\forall i \in N^\rho$, if $\theta_i \leq \tilde{\theta}_\rho^\Gamma - \eta$ then $\sigma_i = L$
(β) $\forall i \in N^\rho$, if $\theta_j \geq \tilde{\theta}_\rho^\Gamma + \eta$ then $\sigma_j = R$.

Proof. Fix η and in Proposition 4, take $\varepsilon = \frac{\eta}{2}$. For the corresponding n_0 it is easy to see that if $n \geq n_0$ and σ is a Nash equilibrium of Γ, then $\tilde{\theta}_\rho^\Gamma - \frac{\eta}{2} \leq X(\bar{\mu}^\sigma) \leq \tilde{\theta}_\rho^\Gamma + \frac{\eta}{2}$. In fact, suppose by contradiction that $\tilde{\theta}_\rho^\Gamma - \frac{\eta}{2} > X(\bar{\mu}^\sigma)$. Proposition 4 implies that all voters to the right of $\tilde{\theta}_\rho^\Gamma$ vote for the rightist party and hence $\tilde{\theta}_\rho^\Gamma \leq X(\bar{\mu}^\sigma)$, contradicting $\tilde{\theta}_\rho^\Gamma - \frac{\eta}{2} > X(\bar{\mu}^\sigma)$. Analogously for the second inequality. Hence $\tilde{\theta}_\rho^\Gamma - \eta \leq X(\bar{\mu}^\sigma) - \frac{\eta}{2}$ and $\tilde{\theta}_\rho^\Gamma - \eta \geq X(\bar{\mu}^\sigma) + \frac{\eta}{2}$, which, with Proposition 4, complete the proof. ■

Every equilibrium conforms to such a cutpoint, and hence, for n large enough, strategic voters basically vote only for the two extremists parties.

4 Example

We now present a very simple example in order to understand how the behavior of ideological voters affects the policy outcome. Moreover, the below example is suitable to study the two different cases, i.e. with and without ideological voters.

Let the distribution function of voters' bliss points be such that at any bliss point there are two voters.

First, let us consider the case when everybody is strategic. The cutpoint policy outcome in the case when everybody is "rational" is (see De Sinopoli and Iannantuoni, 2007):

$$\tilde{\theta}_{\rho}^{\Gamma} \in \zeta_L \bar{H}^{\rho}\left(\tilde{\theta}_{\rho}^{\Gamma}\right) + \zeta_R\left(1 - \bar{H}^{\rho}\left(\tilde{\theta}_{\rho}^{\Gamma}\right)\right).$$

Let now consider the case when one of the two player in each bliss point is strategic while the other is ideological. The cutpoint outcome is the following:

$$\tilde{\theta}_{\rho}^{\Gamma} \in \frac{n^{\rho}}{n^{\rho}+n^{\iota}}\left(\zeta_L \bar{H}^{\rho}\left(\tilde{\theta}_{\rho}^{\Gamma}\right) + \zeta_R\left(1 - \bar{H}^{\rho}\left(\tilde{\theta}_{\rho}^{\Gamma}\right)\right)\right) + \frac{n^{\iota}}{n^{\rho}+n^{\iota}} \sum_{k=1}^{m} \frac{n_k^{\iota}}{n^{\iota}} \zeta_k,$$

which can be rewritten, considering that $n^{\rho} = n^{\iota}$, in the following way:

$$\tilde{\theta}_{\rho}^{\Gamma} \in \frac{1}{2}\left(\zeta_L \bar{H}^{\Gamma}\left(\tilde{\theta}_{\rho}^{\Gamma}\right) + \zeta_R\left(1 - \bar{H}^{\Gamma}\left(\tilde{\theta}_{\rho}^{\Gamma}\right)\right)\right) + \frac{1}{2} \sum_{k=1}^{m} v_k^{\iota} \zeta_k$$

The first term of the right-hand side of the above expression represents the effect on the policy outcome of the strategic voters' behavior. Clearly, this effect is completely analogous to the cutpoint when everybody is strategic, but now weighted by the share of strategic voters. The second term represents the fixed effect of ideological voters' behavior on the outcome.

This example clearly shows that the two cutpoints are not, in general, equal. Suppose that the cutpoint when all voters are strategic is on the right of the cutpoint when ideological voters are present. Hence, there is a subset of strategic voters, those in between the two cutpoints, voting for the leftmost party in order to adjust to the ideological voters' effect. Nevertheless, strategic voters cannot fully adjust.

5 Conclusions

Building on De Sinopoli and Iannantuoni (2007), we have provided a model in which there are policy motivated rational voters who take their voting decision by maximing their utilities, and ideological voters, who simply cast their ballot in favor of the party whose policy is closest to their preferred one. The main question in this context is if ideological voting behavior really matters. The answer is affirmative. We prove that there is "basically" an unique Nash equilibrium characterized by a cutpoint outcome such that any strategic voter on its left votes for the leftmost party and any strategic voter on its right votes for the rightmost party. Moreover, there is a “fixed” effect of the ideological voters’ behavior on the equilibrium outcome for which strategic players cannot fully adjust.

Finally, this extension helps to better reconcile the results of De Sinopoli and Iannantuoni (2007) with the general agreement on the evidence that proportional representation systems are more likely characterized by multi-party (see Cox 1997).

6 Appendix

Proof of proposition 3:

(α) Given a mixed strategy σ_j, the player j's vote is a random vector[7] $\tilde{s}_j$ with $Pr\left(\tilde{s}_j = k\right) = \sigma_j^k$. Given $\sigma_{-i} = (\sigma_1, ...\sigma_{i-1}, \sigma_{i+1}, ...\sigma_n)$, let $\bar{\tilde{s}}^{-i} = \frac{1}{n-1} \sum_{j \in N/i} \tilde{s}_j$ and $\bar{\mu}^{\sigma_{-i}} = \frac{1}{n-1} \sum_{j \in N/i} \sigma_j$. The first step of the proof consists in proving the next lemma:

Lemma 6 *$\forall \phi > 0$ and $\forall \delta > 0$, if $n > \frac{m}{4\phi^2\delta} + 1$, then $\forall \sigma, \forall i$*

$$\Pr\left(\left|\bar{\tilde{s}}^{-i} - \bar{\mu}^{\sigma_{-i}}\right| \leq \vec{\phi}\right) > 1 - \delta.$$

Proof. To prove the lemma we can use Chebichev's inequality component by component. Given σ_{-i}, it is easy to verify that $E(\tilde{s}_j^k) = \sigma_j^k$ and $Var(\tilde{s}_j^k) = \sigma_j^k(1 - \sigma_j^k) \leq \frac{1}{4}$, hence $E(\bar{\tilde{s}}_k^{-i}) = \bar{\mu}_k^{\sigma_{-i}}$ and $Var(\bar{\tilde{s}}_k^{-i}) \leq \frac{1}{4(n-1)}$. By Chebychev's inequality we know that $\forall k, \forall \phi$:

$$\Pr\left(\left|\bar{\tilde{s}}_k^{-i} - \bar{\mu}_k^{\sigma_{-i}}\right| > \phi\right) \leq \frac{1}{4(n-1)\phi^2}.$$

Hence

$$\Pr\left(\left|\bar{\tilde{s}}^{-i} - \bar{\mu}^{\sigma_{-i}}\right| \leq \vec{\phi}\right) \geq 1 - \sum_k \Pr\left(\left|\bar{\tilde{s}}_k^{-i} - \bar{\mu}_k^{\sigma_{-i}}\right| > \phi\right) \geq 1 - \frac{m}{4(n-1)\phi^2},$$

which is strictly greater than $1 - \delta$ for $n > \frac{m}{4\phi^2\delta} + 1$. ■

Lemma 7 *$\forall \varepsilon > 0$, $\exists n_0^L$ such that $\forall n \geq n_0^L$, if the game has n voters and if $\theta_i < X\left(\bar{\mu}^{\sigma}\right) - \varepsilon$, then L is the only best reply for player $i \in N^{\rho}$ to σ^{-i}.*

Proof. Fix $\varepsilon > 0$. Define $\forall \theta \in \left[0, 1 - \frac{\varepsilon}{2}\right]$

$$M_{\varepsilon}\left(\theta\right) = \max_{X \in \left[\theta + \frac{\varepsilon}{2}, 1\right]} \frac{\partial u(X, \theta)}{\partial X}.$$

By single-peakedness we know that $M_{\varepsilon}\left(\theta\right) < 0$. Moreover, given the continuity of $\frac{\partial u(X,\theta)}{\partial X}$ I can apply the theorem of the maximum[8] to deduce that the function

[7] We remind readers that a vote is a vector with m components. Thereafter, given a scalar α, we denote with $\vec{\alpha}$ the vector with m components, all of them equal to α, while given a vector $\beta = (\beta_1, ..., \beta_m)$ with $\mid \beta \mid$ we denote the vector $(\mid \beta_1 \mid, ..., \mid \beta_m \mid)$.

[8] Because there are various versions of the theorem of the maximum, we prefer to state explicitly the version we are using. Let $f : \Psi \times \Phi \to \Re$ be a continuous function and $g : \Phi \to P(\Psi)$ be a compact-valued, continuous correspondence, then $f^*(\phi) := \max\left\{f(\psi, \phi) \mid \psi \in g(\phi)\right\}$ is continuous on Φ.

$M_\varepsilon(\theta)$ is continuous, hence it has a maximum on $\left[0, 1-\frac{\varepsilon}{2}\right]$, which is strictly negative. Let

$$M_\varepsilon^* = \max_{\theta \in \left[0, 1-\frac{\varepsilon}{2}\right]} M_\varepsilon(\theta).$$

Let M denote the upper bound[9] of $\left|\frac{\partial u(X,\theta)}{\partial X}\right|$ on $[0,1]^2$, and let $\delta_\varepsilon^* = \frac{-M_\varepsilon^*}{M - M_\varepsilon^*} > 0$ and $\phi^* = \frac{\left(-2+\sqrt{6}\right)\varepsilon}{m}$. We prove that if $n > \frac{m}{4\phi^{*2}\delta_\varepsilon^*} + 1$, then every strategy other than L cannot be a best reply for player i, which, setting n_0 equal to the smallest integer strictly greater than $\frac{m}{4\phi^{*2}\delta_\varepsilon^*} + 1$, directly implies the claim.[10]

Take a party $c \neq L$. By definition $c \in BR_i(\sigma) \Longrightarrow$

$$\sum_{s_{-i} \in S_{-i}} \sigma(s_{-i}) \left[u\left(X(s_{-i}, c), \theta_i\right) - u\left(X(s_{-i}, L), \theta_i\right)\right] \geq 0, \tag{2}$$

which can be written as:

$$\sum_{s_{-i} \in S_{-i}} \sigma(s_{-i}) \left[u\left(X(s_{-i}, c), \theta_i\right) - u\left(X(s_{-i}, c) - \frac{1}{n}(\zeta_c - \zeta_L), \theta_i\right)\right] \geq 0. \tag{3}$$

Because the outcome function $X(s)$ depends only upon $v(s)$, denoting with V_n^{-i} the set of all vectors representing the share of votes obtained by each party with $(n-1)$ voters, (3) can be written as:

$$\sum_{v_n^{-i} \in V_n^{-i}} \Pr(\bar{\tilde{s}}^{-i} = v_n^{-i}) \left[u\left(X\left(v_n^{-i}, c\right), \theta_i\right) - u\left(X\left(v_n^{-i}, c\right) - \frac{1}{n}(\zeta_c - \zeta_L), \theta_i\right)\right] \geq 0 \tag{4}$$

where, with abuse of notation, $X\left(v_n^{-i}, c\right) = \frac{\zeta_c}{n} + \frac{n-1}{n} \sum_{k=1}^{m} \zeta_k v_{n(k)}^{-i}$. Multiplying both sides of (4) by $\frac{n}{\zeta_c - \zeta_L} > 0$ we have:

$$\sum_{v_n^{-i} \in V_n^{-i}} \Pr(\bar{\tilde{s}}^{-i} = v_n^{-i}) \frac{\left[u\left(X\left(v_n^{-i}, c\right), \theta_i\right) - u\left(X\left(v_n^{-i}, c\right) - \frac{1}{n}(\zeta_c - \zeta_L), \theta_i\right)\right]}{\frac{1}{n}(\zeta_c - \zeta_L)} \geq 0. \tag{5}$$

By the *mean value theorem* we know that $\forall v_n^{-i}$,

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[9] The continuity of $\frac{\partial u(X,\theta)}{\partial X}$ assures that such a bound exists.

[10] This is the same bound we found without ideological voters. Because if j is a ideological player $Var(\tilde{s}_j^k) = 0$, we have that the variance of $\bar{\tilde{s}}_k^{-i}$ decreases with ideological voters, we could perhaps find a better bound. As a matter of fact if $\frac{(n-1)^2}{n^\rho - 1} > \frac{m}{4\phi^2\delta}$ then

$$\Pr\left(\left|\bar{\tilde{s}}^{-i} - \bar{\mu}^{\sigma_{-i}}\right| \leq \vec{\phi}\right) > 1 - \delta.$$

However a preliminary cost-benefit analysis discouraged us from such a project.

$\exists X^* \in \left[X\left(v_n^{-i}, c\right) - \frac{1}{n}(\zeta_c - \zeta_L), X\left(v_n^{-i}, c\right)\right]$ such that

$$\frac{\left[u\left(X\left(v_n^{-i}, c\right), \theta_i\right) - u\left(X\left(v_n^{-i}, c\right) - \frac{1}{n}(\zeta_c - \zeta_L), \theta_i\right)\right]}{\frac{1}{n}(\zeta_c - \zeta_L)} = \left.\frac{\partial u(X, \theta_i)}{\partial X}\right|_{X = X^*}.$$

Hence we have:

$$\sum_{v_n^{-i} \in V_n^{-i}} \Pr(\bar{\tilde{s}}^{-i} = v_n^{-i}) \frac{\left[u\left(X\left(v_n^{-i}, c\right), \theta_i\right) - u\left(X\left(v_n^{-i}, c\right) - \frac{1}{n}(\zeta_c - \zeta_L), \theta_i\right)\right]}{\frac{1}{n}(\zeta_c - \zeta_L)} \leq$$

$$\Pr(\left|\bar{\tilde{s}}^{-i} - \bar{\mu}^{\sigma_{-i}}\right| \leq \vec{\phi}^*) M_n^*(\vec{\phi}^*, \theta_i) + (1 - \Pr(\left|\bar{\tilde{s}}^{-i} - \bar{\mu}^{\sigma_{-i}}\right| \leq \vec{\phi}^*))M$$

where

$$M_n^*(\vec{\phi}^*, \theta_i) = \max_{X \in \left[X(\bar{\mu}^{\sigma_{-i}} - \vec{\phi}^*, c) - \frac{1}{n}(\zeta_c - \zeta_L), 1\right]} \frac{\partial u(X, \theta_i)}{\partial X}.$$

Now we prove that, for $n > \frac{m}{4\phi^{*2}\delta_\varepsilon^*} + 1$, $M_n^*(\vec{\phi}^*, \theta_i) \leq M_\varepsilon^*$. From the definition of M_ε^*, it suffices to prove that $M_n^*(\vec{\phi}^*, \theta_i) \leq M_\varepsilon(\theta_i)$, which is true if $X(\bar{\mu}^{\sigma_{-i}} - \vec{\phi}^*, c) - \frac{1}{n}(\zeta_c - \zeta_L)$ is greater than $\theta_i + \frac{\varepsilon}{2}$.

$$X(\bar{\mu}^{\sigma_{-i}} - \vec{\phi}^*, c) - \frac{1}{n}(\zeta_c - \zeta_L) = \frac{n-1}{n} \sum_k \bar{\mu}_k^{\sigma_{-i}} \zeta_k - \frac{n-1}{n} \sum_k \phi^* \zeta_k + \frac{1}{n}\zeta_L =$$

$$X(\bar{\mu}^\sigma) - \frac{1}{n} \sum_k \sigma_i^k \zeta_k + \frac{1}{n}\zeta_L - \frac{n-1}{n} \sum_k \phi^* \zeta_k >$$

$$X(\bar{\mu}^\sigma) - \frac{1}{n}(\zeta_R - \zeta_L) - m\phi^* \zeta_R \geq \theta_i + \varepsilon - \frac{1}{n} - m\phi^*.$$

Hence this step of the proof is concluded by noticing that δ_ε^* is by definition less than $\frac{1}{2}$, hence[11]

$$\theta_i + \varepsilon - \frac{1}{n} - m\phi^* > \theta_i + \varepsilon - \frac{2\phi^{*2}}{m} - m\phi^* =$$

$$\theta_i + \varepsilon - \frac{(20 - 8\sqrt{6})\varepsilon^2}{m^3} - \varepsilon\left(-2 + \sqrt{6}\right) \geq \theta_i + \varepsilon(1 - \frac{(20 - 8\sqrt{6})}{8} + 2 - \sqrt{6}) =$$

$$\theta_i + \frac{1}{2}\varepsilon.$$

By Lemma 6, we know that, for $n > \frac{m}{4\phi^{*2}\delta_\varepsilon^*} + 1$,

$$\Pr(\left|\bar{\tilde{s}}^{-i} - \bar{\mu}^{\sigma_{-i}}\right| \leq \vec{\phi}^*) M_n^*(\vec{\phi}^*, \theta_i) + (1 - \Pr(\left|\bar{\tilde{s}}^{-i} - \bar{\mu}^{\sigma_{-i}}\right| \leq \vec{\phi}^*))M <$$

[11] In the following we assume that $\varepsilon \leq 1$, since otherwise the proposition is trivially true.

$$(1-\delta_\varepsilon^*)M_\varepsilon^* + \delta_\varepsilon^* M = (1 - \frac{-M_\varepsilon^*}{M - M_\varepsilon^*})M_\varepsilon^* + \frac{-M_\varepsilon^*}{M - M_\varepsilon^*}M = 0.$$

Summarizing, we have proved that for $n > \frac{m}{4\phi^{*2}\delta_\varepsilon^*} + 1$, for every strategy $c \neq L$

$$\sum_{v_n^{-i} \in V_n^{-i}} \Pr(\bar{\tilde{s}}^{-i} = v_n^{-i}) \frac{\left[u\left(X\left(v_n^{-i}, c\right), \theta_i\right) - u\left(X\left(v_n^{-i}, c\right) - \frac{1}{n}(\zeta_c - \zeta_L), \theta_i\right)\right]}{\frac{1}{n}(\zeta_c - \zeta_L)} \leq$$

$$\Pr(\left|\bar{\tilde{s}}^{-i} - \bar{\mu}^{\sigma_{-i}}\right| \leq \vec{\phi}^*) M_n^*(\vec{\phi}^*, \theta_i) + (1 - \Pr(\left|\bar{\tilde{s}}^{-i} - \bar{\mu}^{\sigma_{-i}}\right| \leq \vec{\phi}^*))M <$$

$$(1-\delta_\varepsilon^*)M_\varepsilon^* + \delta_\varepsilon^* M = 0,$$

which implies that c is not a best reply for player $i \in N^\rho$. ■

Analogously, it can be proved the following Lemma.

Lemma 8 *$\forall \varepsilon > 0$, $\exists n_0^R$ such that $\forall n \geq n_0^R$, if the game has n voters and if $\theta_i \geq X(\bar{\mu}^\sigma) + \varepsilon$, then R is the only best reply for player i to σ^{-i}.*

Setting $n_0 = \max\left\{n_0^L, n_0^R\right\}$ completes the proof.

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