The effect of externalities aggregation on network games outcomes

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Abstract

We generalize results on the monotocity of equilibria for network games with incomplete information. In those games players know the stochastic process of network formation and their own degree in the realized network, and decide an action depending on the strategic interaction in the network between their own action and a statistic (as the mean, the maximum or the minimum) of neighbors' actions. We show that, even under degree independence, not only the distinction between *strategic complements* and *strategic substitutes* is important in determining the nature of Bayesian Nash equilibria, but also the nature itself of the statistic.

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1 Introduction

Following the paper "Network Games" by Galeotti et al. (2010) (hencefort: NG), many recent models on games with local externalities assume that agents are nodes of a network environment, and that they have to take an action which has local externalities channeled trough the topology of the network. However, the agents have limited observability on the structure of the network and even on the identity of their peers. Essentially, nodes know only

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their own degree and have some information about the general network formation process that generated the whole social network. The realization could be such that the degree of neighbors is i.i.d., and independent also on the degree of the node itself, as is the case for example of the recent papers by Galeotti and Rogers (2013) and Konno (2013). However, more generally, any stochastic process for the realization of the network could be considered, as is formalized in the recent paper by Acemoglu et al. (2013).

In some sense the nodes, in these *network games* with limited observability, compute *expected statistics* over the sample that they may end up finding in the pool.

From an applied point of view, such models are a good tool for analyzing many complex social phenomena, as the spread of habits, marketing for goods with externalities, vaccination policies, and public good contributions, just to name a few. So, this approach is consistent with the empirical literature on peer effects and reference groups, on which a good survey is given by Blume et al. (2010). However, it must be noted that the theoretical predictions of these models are unclear when it comes to assign some correlation between the degree of nodes and their action: who will endogenously tend to vaccinate more during a flu pandemic, those with many or those with few links (see Goyal and Vigier, 2010, and again Galeotti and Rogers, 2013 and Acemoglu et al., 2013 for models and discussion for this environment)? when a bad habit as smoking spreads in a school, and there are peer effects, who will be more likely to smoke, those with many or those with few links (see Currarini et al. 2013)? As an example of this discordance in the theoretical models, Proposition 10 in NG gives different predictions from the results of Galeotti and Rogers (2013) and Konno (2013). The technical reason is that they are based on slightly different assumptions about the way externalities coming from the neighbours are aggregated. So, it becomes important from a theoretical point of view to study more generally the effects of these kinds of assumptions.

2 The model

Let $\mathcal{N} = \{1, 2, ...n\}$ be a finite set of agents. Each agent $i \in \mathcal{N}$ obtains some partial information about the realization of a random network and then chooses an action $x_i \in \mathcal{X}$, where $\mathcal{X} \subseteq \mathbb{R}$ is a compact set. Payoffs are assigned in a way that depends on the realized network environment. The structure is the one of a Bayesian game, and we follow the notation of NG, integrating it with some of the formalization from Acemoglu et al. (2013).

Network: Our realized network environment is represented by a (possibly directed) *network* g, in which the set of nodes is the set of agents, and a link ij denotes that the action of agent j affects i's payoff. Network g is obtained from a probability distribution P over all the $2^{n(n-1)}$ possible networks. We call P the *network formation process*. Let us call K the set of natural numbers $\{1, 2, \ldots, n-1\}$. We denote by $N_i(g)$ the set of neighbors of i in g (excluding i) and we denote by $k_{i,g} \in K$ the number of such neighbors (i.e. i's out–degree in the network). We call $V_{k,g}$ the set of nodes that have degree k in network g, and by $v_{k,g}$ the cardinality of this set.

Before going on, let us consider as benchmarks two simple probability distributions P. One follows the Erdös and Rényi (1960) model of random networks, where every link between any two nodes is in place with some i.i.d. distribution which is common knowledge. The other is a model of random networks where all and only the networks with a pre–specified degree distribution have positive probability, and this degree distribution is the common prior. In these simple cases, knowing own degree provides no additional information on the degree of neighbors. Following NG, we call *degree independence* this lack of correlation. When instead, knowing own degree k changes the expectation on the degree of neighbors, we may have degree *assortativity* or *disassortativity* (more on this at end of Section 3).

Payoffs: Payoffs are based on the realized network g. Player i's payoff function when she chooses x_i and her $k_{i,g}$ neighbors choose the action profile $\vec{x}_{i,g} = (x_1; ...; x_{k_i})$ is:

$$\Pi_{k_{i,g}}(x_i, \vec{x}_{i,g}) = f\left(x_i, s(\vec{x}_{i,g})\right) - c(x_i) \quad , \tag{1}$$

where s is a measure computed on the set of the neighbors' actions,[1] and $f(x,s)$ is continuously differentiable function from $\mathbb{R}^2$ to $\mathbb{R}$ with:

$$f_x > 0 \ , \quad f_s > 0 \ , \quad f_{xx} \leq 0 \quad .$$

If either $f_{xs} \leq 0$ or $f_{xs} \geq 0$, we say, respectively, that f has the *substitutes* or the *complements* property. Finally, $c(x)$ is a convex cost function such that $c_x > 0$ and $c_{xx} > 0$.

Statistic: The effects of local interaction are aggregated by the *measure* s. Formally s is a different k–dimensional function for every $k \in K$. So, s is a family of $n-1$ functions[2] and each of them is anonymous on the arguments, which means that any permutation of the elements of the sample $\vec{x}_{i,g}$ would give the same result, for any size of $\vec{x}_{i,g}$. This very general specification includes measures of central tendency as well as measures of variability

[1] Formally, s is a function from $\mathbb{R}^{k_{i,g}}$ to $\mathbb{R}$.

[2] In principle we allow them to be even different functions for each k

(or dispersion). Measures of central tendency include the mean, median and mode, while measures of variability include the standard deviation (or variance), the minimum and maximum values of the variables, kurtosis and skewness. In the following we refer to any measure s as a *statistic*.

Information: The only piece of information that an agent i obtains before deciding her action, on top of the common prior P, is her own degree k_i in some realized network g. Then players play a game of incomplete information described by the quadruple $(N, \mathcal{X}, (\Pi_k)_{k\in K}, P)$.

Bayesian game: A strategy for player i is a mapping $\sigma_i : K \to \Delta(\mathcal{X})$, where $\Delta(\mathcal{X})$ is the set of probability distributions on $\mathcal{X}$, i.e. $\sigma_i = [\sigma_{ik}]_{k\in K}$ where σ_{ik} is the mixed strategy played by player i of degree k. Furthermore $\vec{\sigma}_{ig}$ is the strategy profile of i's neighbors in network g, $\vec{\sigma} = [\sigma_i]_{i\in N}$ is the strategy profile of the game and $\vec{\sigma}_{-i} = [\sigma_j]_{j\in N/i}$ is the set of strategy profiles of all players excluded i.
We consider *symmetric Bayesian Nash equilibria* in which every agent with the same information and facing the same ex–ante conditions (i.e. each agent i with the same degree k) chooses the same strategy, i.e. $\sigma_{ik} = \sigma_{jk}\ \forall\ k \in K$ and $\forall\ i, j \in N$.
We say that a strategy profile $\vec{\sigma}$ is *first order stochastic dominance (FOSD) increasing* if, for every $k \in K\backslash\{n-1\}$ we have that σ_{k+1} FOSD σ_k (which is to say that the cumulative distribution of σ_{k+1} is always below the cumulative distribution of σ_k – in the context of pure strategies it means that $x_{k+1} \geq x_k$). Analogously, $\vec{\sigma}$ is *FOSD decreasing* if, for every $k \in K\backslash\{n-1\}$ we have that σ_k FOSD σ_{k+1}.

Given a realized network g and a strategy profile $\vec{\sigma}$ the expected payoff of agent i of degree k, who know her position and the positions of all other nodes, is given by:

$$\Pi^e_{i,g}(\sigma_i, \vec{\sigma}_{ig}) = \int_{\mathcal{X}^N} \Pi_{k_{i,g}}(x_i, \vec{x}_{i,g})\ d\vec{\sigma} \quad . \tag{2}$$

On this, we have to include also the uncertainty about the realization of the network. Adding this, the expected payoff of agent i of degree k is

$$\Pi^e_k(\sigma_i, \vec{\sigma}_{-i}) = \frac{\sum_g P(g) \cdot \sum_{i\in V_{k,g}} \Pi^e_{i,g}(\sigma_i, \vec{\sigma}_{ig})}{\sum_g P(g) \cdot v_{k,g}} \quad . \tag{3}$$

In words, an agent evaluates all possible nodes i with degree k in any possible realized network g, updating priors with the information that her degree is actually k.[3] For each

[3]Note that $\Pi^e_k(\sigma_i, \vec{\sigma}_{-i}) = \sum_g P(g|k) \cdot \frac{\sum_{i\in V_{k,g}} \Pi^e_{ig}(\sigma_i, \vec{\sigma}_{ig})}{v_{k,g}}$ where $P(g|k) = \frac{v_{k,g}\cdot P(g)}{\sum_g v_{k,g}\cdot P(g)}$ is the update probability that network g is in force.

such node position i and network g, and for each realization of $\vec{\sigma}$, there will be a vector $\vec{x}_{i,g}$ that lists the action of each neighbor, depending on their degree in network g.

The Bayesian Nash equilibria can be represented simply as a (mixed) strategy profile $\vec{\sigma}^*$, where every agent i, depending on her degree k_i, will choose an optimal strategy σ_k^*, that maximizes the individual expected payoff for agent i from (3).
Let $\phi_{ig}(s|\vec{\sigma}_{ig})$ be the probability density function of s exactly for node i in network g when the strategy profile of the i's neighbors is $\vec{\sigma}_{ig}$. For an agent observing only her own degree k the posterior distribution for the statistic s will be:

$$\phi_k(s|\vec{\sigma}, P) \equiv \frac{\sum_g P(g) \cdot \sum_{i \in V_{k,g}} \phi_{ig}(s|\vec{\sigma}_{ig})}{\sum_g P(g) \cdot v_{k,g}} \quad , \tag{4}$$

Therefore, since the Bayesian updating based on the network structure is linear, the expected value of s for an agent of degree k is

$$E_k(s|\vec{\sigma}, P) = \frac{\sum_g P(g) \cdot \sum_{i \in V_{k,g}} E_{ig}\left(s|\vec{\sigma}_{ig}\right)}{\sum_g P(g) \cdot v_{k,g}} \quad , \tag{5}$$

where $E_{ig}\left(s|\vec{\sigma}_{ig}\right)$ is the expected value of s for node i in network g and when the strategy profile of the i's neighbors is $\vec{\sigma}_{ig}$.
We call $\Phi_k(s|\vec{\sigma}, P)$ the cumulative probability distribution on s from $\phi_k(s|\vec{\sigma}, P)$. Then $\Phi_k(s|\vec{\sigma}, P)$ summarizes all the information provided by P (the network formation process) and $\vec{\sigma}$ (the strategy profile). Finally the variance of s for an agent of degree k when the strategy profile of the game is $\vec{\sigma}$ and the network formation process is P is denoted by $Var_k(s|\vec{\sigma}, P)$

Definitions: Given that the type of statistic $s(.)$ affects the individual payoff and subsequently the optimal individual behavior, through the network formation process P and equilibrium strategy $\vec{\sigma}$, we highlight its relevant characteristics.

Definition **1.** *Let P have degree independence. A statistic s is* stable *if for every $\vec{\sigma}$ and $k \in K\backslash\{n-1\}$ we have that $E_{k+1}(s|\vec{\sigma}, P) = E_k(s|\vec{\sigma}, P)$.*

Definition **2.** *Let P have degree independence. A statistic s is* increasing *(or* decreasing*) if for every $\vec{\sigma}$ and $k \in K\backslash\{n-1\}$ we have that $E_{k+1}(s|\vec{\sigma}, P) > E_k(s|\vec{\sigma}, P)$ (or respectively, if for every $\vec{\sigma}$ and $k \in K\backslash\{n-1\}$ we have that $E_{k+1}(s|\vec{\sigma}, P) < E_k(s|\vec{\sigma}, P)$).*

Definition **3.** *Let P have degree independence. A statistic s is* FOSD increasing *(or* FOSD decreasing*) if for every $\vec{\sigma}$, $k \in K\backslash\{n-1\}$, and $x \in \mathbb{R}$, we have that $\Phi_{k+1}(x|\vec{\sigma}, P) \leq$*

$\Phi_k(x|\vec{\sigma}, P)$ (or respectively, if for every $\vec{\sigma}$, $k \in K \backslash \{n-1\}$, and $x \in \mathbb{R}$, we have that $\Phi_{k+1}(x|\vec{\sigma}, P) \geq \Phi_k(x|\vec{\sigma}, P)$). We say that statistics s is strictly FOSD increasing (or decreasing) when all inequalities are stricty satisfied.

DEFINITION **4.** *Let P have degree independence. A statistic s satisfies second order stochastic dominance (SOSD): if for every $\vec{\sigma}$ and $y \in \mathbb{R}$ we have the following inequality:*

$$\int_{-\infty}^{y} \Phi_{k+1}(x|\vec{\sigma}, P)\ dx \leq \int_{-\infty}^{y} \Phi_k(x|\vec{\sigma}, P)\ dx \quad . \tag{6}$$

The following two results could be useful to understand the meaning of the assumptions made in our propositions.. First, it is directly verifiable tht a strictly FOSD increasing statistic (Definition 3) implies an increasing one (Definition 2). Second, if s is stable and satisfies SOSD (Definitions 1 and 4), then s is *converging*, in the sense that for every σ and $k \in K \backslash \{n-1\}$, we have that $Var_{k+1}(s|\vec{\sigma}, P) < Var_k(s|\vec{\sigma}, P)$ (we prove this in Appendix A as a corollary to a lemma).

When P has degree independence, for any strategy profile $\vec{\sigma}$, many standard statistics as the mean, the median, or the sample variance, are both stable and converging. Still under degree independence, examples of increasing and decreasing statistics are instead, respectively, the maximum and the minimum (whenever the strategy profile $\vec{\sigma}$ is not the same constant for each $k \in K$).[4]

3 Results

In this game context the existence of a symmetric Bayesian Nash equilibrium follows directly from Kakutani fixed point theorem, as mixed equilibria on a compact set $\mathcal{X}$ form themselves a convex compact set.
Our main result is about the characterization of Bayesian Nash equilibria in the case P has degree independence.

PROPOSITION **1.** *Let the network formation process P be characterized by degree independence, then:*

[4]Minimum and maximum contribution from neighbors relate respectively to *weakest–link* and *best–shot* games, as introduced by Hirshleifer (1983) in a non–network context. NG and Boncinelli and Pin (2012) discuss network best shot games, while classical *weakest–link* games are those related to contagion, as Galeotti and Rogers (2013). On this see also the discussion in Jackson and Zenou (2014) about *strategic complements* and *strategic substitutes*. We show in next Proposition 1, however, that a more important distinction is in the monotonicity of the statistic.

1. *if s is stable and satisfies SOSD, then for any symmetric Bayesian Nash equilibrium σ^*, (i) if $f_{xss} > 0$ then σ^* is FOSD decreasing, if (ii) $f_{xss} < 0$ then σ^* is FOSD increasing, if finally (iii) $f_{xss} = 0$ then $\sigma^*_k = \sigma^*_{k'}$ for each $k, k' \in K$;*

2. *if s is FOSD increasing, then for any symmetric Bayesian Nash equilibrium σ^*, (i) if $f_{xs} > 0$ then σ^* is FOSD increasing, if instead (ii) $f_{xs} < 0$ then σ^* is FOSD decreasing;*

3. *if s is FOSD decreasing, then for any symmetric Bayesian Nash equilibrium σ^*, (i) if $f_{xs} > 0$ then σ^* is FOSD decreasing, if instead (ii) $f_{xs} < 0$ then σ^* is FOSD increasing.*

The formal proof of Proposition 1 is in Appendix B, as all the proofs of following results. Point (1) follows directly from a result that we show in Lemma 1 in Appendix A. The proof of Points (2) and (3) is instead analogous to the proof of Proposition 2 in NG.

Note that the assumptions on the properties of statistic s are not so restrictive. Indeed many of the statistics used in the literature of strategic interaction satisfy these assumption. Ideally, we would like to take the less restrictive assumption of SOSD in points (2) and (3) but it requires to have more assumptions on the shape of the function f.

From the analytical derivations obtained in the proofs, it is easy to extend our results to the cases where P has not degree independence. In the literature on complex networks, stemming from Newman (2002), a network exhibits degree *assortativity* or *disassortativity*, depending on the sign of the Pearson degree of correlation between degrees in the network. However, in NG this notion is related to the function that rules the network externalities of the game (i.e. f), and they talk about *positive* or *negative neighbour affiliation.* To obtain a general result, we also use an inequality that aggregates the function f that determines payoffs with all the information that we have about the network structure and the statistics from $\phi_k(y|\vec{\sigma}, P)$.

PROPOSITION **2.** *Consider the expected marginal profits given by quantity*

$$E_k(f_x|x, \vec{\sigma}, P) \equiv \int_{-\infty}^{\infty} f_x(x, y) \cdot \phi_k(y|\vec{\sigma}, P)\, dy \tag{7}$$

(where f_x is the derivative of f with respect to x).
We have the following:

1. *if the quantity in (7) is strictly decreasing in k for any $\vec{\sigma}$, then in every symmetric Bayesian Nash equilibrium of the network game the optimal action σ^*_k is FOSD decreasing in k;*

2. *if instead the quantity in (7) is strictly increasing in k for any $\vec{\sigma}$, then in every symmetric Bayesian Nash equilibrium of the network game the optimal action σ_k^* is FOSD increasing in k.*

Proposition 2 provides a general check to see if there is monotonicity in the equilibrium of the game. Figure 1 provides a visual interpretation of the argument in the proof, which is classical: from the payoff function (1) the optimal response for a player is when marginal costs (increasing by assumptions) intersect expected marginal profits (decreasing by assumption) – so, if an increase/decrease in k has a monotonic effect on those expected marginal profits, also the intersection point will move monotonically.

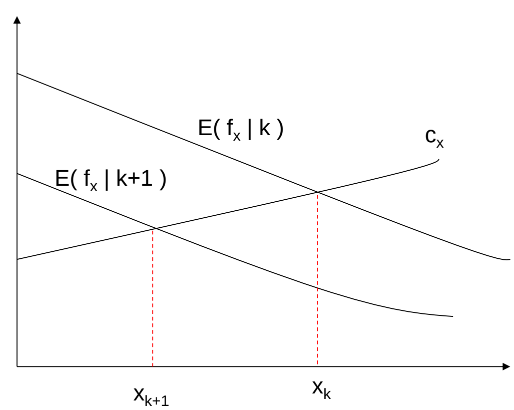

Figure 1: Intuition for Proposition 2 when $x^*(k) \geq x^*(k+1)$.

From the next example it is clear that the standard distinction between complements and substitute is not enough to determine the characterization of equilibria.

Example **1.** *Consider P such that networks are undirected, nodes can have only degree 1 or 2, and they face ex–ante symmetric probability $0 < p < 1$ to find all neighbors of the same degree, and $1 - p$ of finding all neighbors of the other degree (so, when $p \to 1$ we have a network made almost only by circles and disconnected couples, as p decreases we have more* *and more triplets of nodes in a line). Consider for simplicity that nodes play symmetric pure strategies $y_1, y_2 \in \mathcal{X} \subseteq \mathbb{R}_+$, and that the statistic s is just the sum of neighbors' actions. Proposition 2 tells us that we need to consider the relation between*

$$E_1(f_x|x, \vec{\sigma}, P) = pf_x(x, y_1) + (1-p)f_x(x, y_2)$$

and

$$E_2(f_x|x, \vec{\sigma}, P) = pf_x(x, 2y_2) + (1-p)f_x(x, 2y_1) \quad .$$

Assume that $1/3 < p < 2/3$, so that $2p > (1-p)$ and $2(1-p) > p$. Assume also that f_x is increasing (complementarity between x and s) and convex ($f_{xss} > 0$) in s, so that $f_x(x, 2s) - f_x(x, 0) > 2\Big(f_x(x, s) - f_x(x, 0)\Big)$. Then, for any $y_1, y_2 \geq 0$, the following holds

$$\begin{aligned}
E_2(f_x|x, \vec{\sigma}, P) &= p\Big(f_x(x, 2y_2) - f_x(x, 0)\Big) + (1-p)\Big(f_x(x, 2y_1) - f_x(x, 0)\Big) + f_x(x, 0) \\
&\geq 2p\Big(f_x(x, y_2) - f_x(x, 0)\Big) + 2(1-p)\Big(f_x(x, y_1) - f_x(x, 0)\Big) + f_x(x, 0) \\
&> (1-p)\Big(f_x(x, y_2) - f_x(x, 0)\Big) + p\Big(f_x(x, y_1) - f_x(x, 0)\Big) + f_x(x, 0) \\
&= E_1(f_x|x, \vec{\sigma}, P)
\end{aligned}$$

So, according to Proposition 2, in every symmetric Bayesian Nash equilibrium of the network game, optimal best responses are such that $x_1^* < x_2^*$. ▮

In the next section we discuss more in depth the relation between our result and those in the previous literature.

4 Relation with previous literature

The results provided in Propositions 1 and 2 include all the cases that we are aware of in the literature on Bayesian network games. In particular, they generalize those of NG in several ways. First, they depend on whatever *statistics* that enters in the strategic interaction, and on its relation with the game structure. Then, the classical distinction between substitutability and complementarity holds only when the statistic is naturally monotonic, otherwise we need to check for third cross-derivatives or for monotonicity of expected marginal profits. Finally, even if all the results of Proposition 1 are proved in the case where the network structure P has degree independence, Proposition 2 allows to extended them when some degree assortativity/disassortativity of the network is present (and as Example 1 shows, it may be possible only when this network bias is not too strong).

So, what are the analogies with NG? Most of the results in that paper (from Proposition 2 on) are based on what they call Property A: if the statistic is applied to a new vector obtained from the old one adding a null element, then the statistic provides the same value.[5] Also, Property A alone is not enough to guarantee a monotonic equilibrium in the context of our generalized model. The following example shows that if s is not monotonic with respect to its arguments, we may not have monotonic equilibria. Then we show that, taking that monotonicity into account, property A leads to the case of FOSD increasing statistics.

Example **2.** *Consider the case in which $\mathcal{X} = \{0, 1\}$, with the statistic s defined on every vector of at least two elements, as the difference between its two greatest elements. This statistic clearly satisfies Property A from NG. Since $\mathcal{X} = \{0, 1\}$, we have that s is 1 if and only if there is one and only one element 1 in the vector, otherwise it is 0. Consider the case of degree independence, so that the matching process is i.i.d.. So, if a fraction p of the nodes play 1, then the probability that s is 1 is*

$$p_k = k \cdot p(1-p)^{k-1} \quad ,$$

which can be non–monotonic in k. Imagine that the degree distribution is such that a fraction $.15$ of the nodes have degree 2, a fraction $.7$ have degree 3, and the remaining fraction $.15$ of nodes have degree 4. Utilities are of the form $f(x, s) = \sqrt{x+s}$ (a case of substitutes), $c(0) = 0$ and $c(1) = .75$.

In this case there is an equilibrium in which nodes with degree 2 and 4 contributes with 1, while nodes with degree 3 free–ride contributing 0. With this strategy profile $p = .3$, $p_2 = .42$, $p_3 = .441$ and $p_4 = .4116$. The expected net value of contributing is given by

$$\Delta_k = p_k\left(\sqrt{2}-1\right) + (1-p_k) \quad ,$$

and this is above $.75$ for $k \in \{2, 4\}$, but not for $k = 3$, proving that this strategy profile is an equilibrium. ▮

The following result provides the link between our formulation and the results from NG.

Proposition **3.** *Suppose that $\mathcal{X} \subseteq \mathbb{R}_+$, and that $0 \in \mathcal{X}$. Then, if s is monotonic increasing in all its arguments, and satisfies Property A from NG, then it is FOSD increasing.*

Even if in the NG's payoff function there is not an explicit statistic s but only a vector of the neighbors' actions, it is possible to check that the main results in NG are included in point (2) of our Proposition 1.

[5] The easiest example of statistic that does not satisfy this property is the average.

5 Conclusion

In many applications, externalities, peer effects, learning and/or strategic interactions between individuals adopting a specific action, can all be easily modeled as network games between agents of a social networks. The network neighbors of a node are in one to one correspondence with the peers of the individual, and the actions of those neighbors enter in the payoff function of the individual. The way in which they do so can be the average, as in most of peer effects framings or non–Bayesian learning models, the maximum, as in local public goods games, or the minimum, as in vaccination games against the risk of pandemic contagion. The existing literature points out the influence that the nature of this payoff function, and in particular whether there is a complementarity or a substitute effect between own action and the statistic on the actions of neighbors, has on the correlation between degree in the network and action taken in equilibrium. However, in this paper we have shown that in all the above cases it is as important to know also the nature of the statistics.

As an example, vaccination and acquisition of information are both activities with positive externalities and the substitute property (i.e. the more my neighbors contribute the less I will, in equilibrium). However, they differ in the statistic that affects payoffs. In the vaccination case I am influenced by the minimum contribution in my neighborhood (it is a *weakest–link* game), so that having more neighbors increases the probability of finding a non–vaccinated one, and agents with higher degree will be more likely to vaccinate. In the information acquisition case I am influenced by the neighbor who knows more (it is a *best–shot* game), so that having more neighbors increases the probability of finding a well–informed one, and agents with higher degree will be more likely to free ride and not acquire information themselves.

We have also shown that when this statistic is not expected to increase or decrease with the size of the neighbors sample, then it becomes important to look also at third order derivatives of the payoff function. We believe that our results will turn out to be useful for both theorists studying specific models, and for applied researchers studying the interactions of economic agents.

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Appendix A Some Lemmas

We extend results from utility theory (see e.g. section 4.2 in the notes from Levin 2006) to our context with the following corollary.

We define by $\phi_k(x)$ a bounded probability distribution function on $\mathbb{R}$ that depends on $k \in K$, and we call $\Phi_k(x)$ its cumulative distribution. Following Definition 4, we say that $\phi_k(x)$ satisfies second order stochastic dominance (SOSD) if for every $y \in \mathbb{R}$ we have:

$$\int_{-\infty}^{y} \Phi_{k+1}(x)\ dx \leq \int_{-\infty}^{y} \Phi_k(x)\ dx \quad . \tag{8}$$

LEMMA **1.** *If statistic s is stable and satisfies SOSD, and $u(\cdot)$ is a positive valued concave function, then*

$$\int u(s) \cdot \phi_{k+1}(s)\ ds \geq \int u(s) \cdot \phi_k(s)\ ds \quad . \tag{9}$$

If instead $u(\cdot)$ is a positive valued convex function, then

$$\int u(s) \cdot \phi_{k+1}(s)\ ds \leq \int u(s) \cdot \phi_k(s)\ ds \quad . \tag{10}$$

Proof: Let us start by assuming that $\phi_k(x)$ is stable and satisfies SOSD, and that u is positive valued and concave, i.e. that $u > 0$ and $u_{xx} \leq 0$. Let us call $I(x) \equiv \int_{-\infty}^{x} \Phi_k(y)\ dy) - \int_{-\infty}^{x} \Phi_{k+1}(y)\ dy$, which is non–negative by inequality (8). Also, integrating by parts

$$\int_{-\infty}^{x} \Phi_k(y)\ dy = [y \cdot \Phi_k(y)]_{-\infty}^{x} - \int_{-\infty}^{x} y\ d\Phi_k(y)$$

Replacing into the expression for $I(x)$ and taking its limit to ∞, the stability of $\phi_k(x)$ implies that

$$\lim_{x\to\infty} I(x) = \int_{-\infty}^{\infty} y\ d\Phi_{k+1}(y)) - \int_{-\infty}^{\infty} y\ d\Phi_k(y) = 0 \quad . \tag{11}$$

Since $I(x)$ is non–negative, also

$$-\int_{-\infty}^{\infty} u_{xx}(x) I(x) dx \geq 0 \tag{12}$$

Integrating by parts, the left hand side of expression (12) is equivalent to

$$[-u_x(x) \cdot I(x)]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} u_x(x) \Big(\Phi_k(x) - \Phi_{k+1}(x)\Big) dx \tag{13}$$

By (11) the first term is equal to 0. Then again integrating by parts we get

$$\Big[u(x)\Big(\Phi_k(x) - \Phi_{k+1}(x)\Big)\Big]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} u(x)\Big(\phi_k(x) - \phi_{k+1}(x)\Big) dx \tag{14}$$

It is directly verifiable that the first term is equal to 0. Therefore inequality (12) can be rewritten as:

$$-\int_{-\infty}^{\infty} u(x)\Big(\phi_k(x) - \phi_{k+1}(x)\Big)dx \geq 0 \quad , \tag{15}$$

so $\int u(x)\Big(\phi_k(x) - \phi_{k+1}(x)\Big)dy$ is non–positive, which proves the statement. With the same reasoning, the case in which u is positive valued and convex, i.e. that $u > 0$ and $u_{xx} \geq 0$, leads to the reverse inequality. ▮

Corollary **1.** *If statistic s is stable and satisfies SOSD, then $Var_{k+1}(s|\sigma, P) < Var_k(s|\sigma, P)$.*

Proof: We have that

$$Var_k(s|\sigma, P) = \int s^2 \cdot \phi_k(s)\ ds - (E_k(s))^2 \quad .$$

So, when s is stable $(E(s))^2$ remains constant, and since s^2 is convex we get the result from previous Lemma 1. ▮

Lemma **2.** *If the statistic s is FOSD increasing, and $u(\cdot)$ is a positive valued non–decreasing (non–increasing) function, then*

$$\int u(s) \cdot \phi_{k+1}(s)\ ds \geq \Big(\leq \text{, respectively)}\Big) \int u(s) \cdot \phi_k(s))\ ds \quad . \tag{16}$$

If the sample statistic s is FOSD decreasing, and $u(\cdot)$ is a positive non–decreasing (non–increasing) function, then

$$\int u(s) \cdot \phi_{k+1}(s)\ ds \leq \Big(\geq \text{, respectively)}\Big) \int u(s) \cdot \phi_k(s))\ ds \quad . \tag{17}$$

Proof: An equality found in previous proof is that

$$\int_{-\infty}^{\infty} u_x(x)\Big(\Phi_k(x) - \Phi_{k+1}(x)\Big)dy = -\int_{-\infty}^{\infty} u(x)\Big(\phi_k(x) - \phi_{k+1}(x)\Big)dx$$

So, a sufficient condition to determine the sign of

$$\int_{-\infty}^{\infty} u(x)\Big(\phi_{k+1}(x) - \phi_k(x)\Big)dx$$

is the sign of the integral in the left–hand side. When statistc s is FOSD increasing and $u(\cdot)$ is non-decreasing (non-increasing) we have that $\Big(\Phi_k(x) - \Phi_{k+1}(x)\Big) \geq 0$ and $u_x(x) \geq 0$

$(u_x(x) \le 0)$ for every x so the integral on the left hand side is non-negative (non-positive). The second part of the lemma is proved in similar way and it is omitted ▮

Appendix B Proof of the Propositions

We use a reverse approach in proving our propositions. We first prove the technical result of Proposition 2, and then we use it as a lemma to prove Proposition 1. Finally, we prove Proposition 3.

Proof of Proposition 2 (page 7): Suppose that the quantity in (7) is strictly decreasing in k. An agent with degree k compute expected first derivatives, in order to find x_k^* that satisfies first order conditions. That is

$$E\left[\frac{\partial}{\partial x} f\left(x^*, s(\vec{x}_{i,g})\right)\right] = \frac{\partial}{\partial x} c(x^*) \quad ,$$

or equivalently

$$\int_{-\infty}^{\infty} f_x\left(x^*, y\right) \cdot \phi_k(y|\vec{\sigma}, P))\; dy = c_x(x^*) \quad .$$

Since f_x and c_x are both strictly positive, and they are both strictly monotone with different sign, there is a unique $x_k^* \in \mathbb{R}$ that satisfies the equality. If this $x_k^* \in \mathcal{X}$, then $\sigma_k^* = x_k^*$ is a pure strategy. However, this x_k^* could not be an element of $\mathcal{X}$. In this last case the optimal σ_k^* should play one of the two (possibly both), left–most x_k^{*-} or rightmost x_k^{*+}, elements of $\mathcal{X}$ closest to x_k^* in $\mathbb{R}$. If x_k^{*-} and x_k^{*+} give different expected payoffs, then σ_k^* would be a pure strategy playing the best one of the two. Only in the case in which x_k^{*-} and x_k^{*+} give the same payoff, then any randomization σ_k^* between those two points would be an optimal best response.

By assumption, for an agent with degree $k+1$, we have that

$$\int_{-\infty}^{\infty} f_x\left(x^*, y\right) \cdot \phi_{k+1}(y|\vec{\sigma}, P))\; dy < c_x(x^*) \quad . \tag{18}$$

Left–hand part of (18) is decreasing in x, right–hand part is increasing, and then to balance it back we need $x_{k+1}^* < x_k^*$. Equality of best response strategies may hold only when $x_k^* \notin \mathcal{X}$ but not when σ_k^* is a randomization between two points, because if the expected payoff of x_k^{*-} and x_k^{*+} is the same for an agent with degree k, then x_k^{*+} will provide a higher payoff for for an agent with degree $k+1$.

This proves that if the quantity in (7) is strictly decreasing in k, for every $x \in \mathcal{X}$, then in

every symmetric Bayesian Nash equilibrium of the network game the optimal action σ_k^* is FOSD decreasing in k.
The reverse inequality can be proved analogously. ▮

Now we are ready to prove the main proposition.

Proof of Proposition 1 (page 6): Point 1. Let s be stable and converging. The derivative with respect to x_i of i's expected payoff $\frac{\partial}{\partial x_i}\Pi_k^e(x_i, \vec{\sigma}_{-i}) = \int_{-\infty}^{\infty} f_x(x,s) \cdot \phi_k(s|\vec{\sigma}, P)\ ds$. If $f_{xss} > 0$ by Lemma 1 we have that the derivative is decreasing in k. Then, by Proposition 2, it directly follows that σ^* is FOSD decreasing. If $f_{xss} < 0$ by Lemma 1 we have that the derivative is increasing in k. Then, by Proposition 2 directly follows that σ^* is FOSD increasing.

Point 2. Let s be FOSD increasing . The derivative with respect to x_i of i's expected payoff $\frac{\partial}{\partial x_i}\Pi_k^e(x_i, \vec{\sigma}_{-i}) = \int_{-\infty}^{\infty} f_x(x,s) \cdot \phi_k(s|\vec{\sigma}, P)\ ds$. If $f_{xs} \geq 0$ by Lemma 2 we have that the derivative is increasing in k. Then, by Proposition 2 directly follows that σ^* is FOSD increasing. If $f_{xs} \leq 0$ by Lemma 2 we have that the derivative is decreasing in k. Then, by Proposition 2 directly follows that σ^* is FOSD decreasing.

Point 3. Let s be FOSD decreasing . The derivative with respect to x_i of i's expected payoff $\frac{\partial}{\partial x_i}\Pi_k^e(x_i, \vec{\sigma}_{-i}) = \int_{-\infty}^{\infty} f_x(x,s) \cdot \phi_k(s|\vec{\sigma}, P)\ ds$. If $f_{xs} \geq 0$ by Lemma 2 we have that the derivative is decreasing in k. Then, by Proposition 2 directly follows that σ^* is FOSD decreasing. If $f_{xs} \leq 0$ by Lemma 2 we have that the derivative is increasing in k. Then, by Proposition 2 directly follows that σ^* is FOSD increasing. ▮

Finally, we prove Proposition 3, that relates our result with those in NG.

Proof of Proposition 3 (page 10): For every y in $\mathbb{R}$ we have that

$$\Phi[y|s,k] = Prob\left[\vec{x} \in \mathcal{X}^k : s(\vec{x}) \leq y\right] \quad ,$$

and

$$\Phi[y|s,k+1] = Prob\left[\vec{x} \in \mathcal{X}^{k+1} : s(\vec{x}) \leq y\right] \quad ,$$

Consider the operator $\sigma_0 : \mathcal{X}^{k+1} \to \mathcal{X}^{k+1}$ that takes a random element of $\vec{x}$ (with uniform probabilities) and puts it to 0. Then, also $E\left[s \circ \sigma_0(\cdot)\right]$ is a statistic (as it is anonymous), and by monotonicity of s it is always the case that $s(\vec{x}) \geq E\left[s \circ \sigma_0(\vec{x})\right]$. Note also that it is probabilistically the same to extract with uniform probabilities k elements, or to extract $k+1$ elements, and then remove randomly one of them. So, we have that for every y

$$\Phi[y|s,k+1] \leq \Phi[y|E\left[s \circ \sigma_0(\cdot)\right], k+1] = \Phi[y|s,k] \quad ,$$

which proves the statement. ∎