EXcess Idle Time*

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Abstract

We introduce a novel economic indicator, named *excess idle time* (EXIT), measuring the extent of sluggishness in observed financial prices. Using a complete limit theory and formal tests, we provide econometric support for the fact that high-frequency transaction prices are, coherently with liquidity and asymmetric information theories of price determination, generally *stickier* than implied by the ubiquitous semimartingale assumption and its noise-contaminated counterpart. EXIT provides, for every asset and each trading day, an effective proxy for the extent of illiquidity which is easily implementable using transaction prices only. When applied to the market, EXIT uncovers an economically-meaningful short-term and long-term compensation for illiquidity risk in market returns.

Keywords: Liquidity, asymmetric information, liquidity premium, continuous-time semimartingales, infill asymptotics.

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1 Introduction

If financial prices are Ito semimartingales, high-frequency returns should be above an appropriately-defined shrinking threshold with large probability. In other words, a large incidence of small returns is in contradiction with semimartingale-type behaviour.

Consistent with this intuition, we introduce a novel stochastic quantity, named *idle time*. When computed over a trading day, *idle time* yields the fraction of a day for which price adjustments are sluggish. Such a fraction converges to zero if the price process is an Ito semimartingale, but tends to a number between 0 (not included) and 1 if the price dynamics are staler than implied by the ubiquitous semimartingale assumption. Formally, define by $p_{t_0}, \ldots, p_{t_n}$ the (logarithmic) transaction prices over the time period $[0, T]$. Let ξ_n be a vanishing real sequence. *Idle time* is defined as

$$\mathrm{IT} = \frac{1}{T}\sum_{i=1}^{n}(t_i - t_{i-1})1_{\{|p_{t_i} - p_{t_{i-1}}| \leq \xi_n\}}. \tag{1.1}$$

We derive the distributional limiting properties of *idle time* and its bias-corrected counterpart, *excess idle time* or EXIT, using an asymptotic design which increases the number of price observations over a trading period ($n \to \infty$). Under the semimartingale null hypothesis and suitable assumptions on the threshold, we show that the asymptotic distribution of EXIT is (stably, mixed) normal, centered at zero, and with a limiting variance which depends inversely on the infinitesimal (spot) variance of the return process. A more volatile return process would therefore lead to a more concentrated distribution around zero and stronger rejections of the semimartingale null.

What leads to price staleness? Classical models of price formation postulate that informed traders react to new information not yet reflected in the price of a security, and transact, only if the trade guarantees a profit net of transaction costs. Consequently, due to lack of trading, a security with higher transaction costs should experience less frequent price updates and a larger number of "small" returns than a security with a lower cost of transacting. The incidence of "small returns" can therefore be viewed as being correlated with the magnitude of transaction costs as well as with the probability of arrival of informed traders. Given a certain probability of arrival of informed traders, the smaller the transaction cost, the stronger their ability to exploit deviations between equilibrium prices and observed prices, the lower the number of "small" price updates. Conversely, given a certain size of transaction costs, the higher the probability of arrival of informed traders, the higher - in general - the likelihood of sluggish price adjustments.

This logic, grounded in adverse selection models of price determination as in Glosten and Milgrom (1985) and Kyle (1985) among others, clarifies that there exists an important interaction between liquidity and asymmetric information. Since the size of transaction costs is correlated with liquidity, the number of stale price adjustments (and, consequently, the magnitude of EXIT) will grow both with increases in illiquidity and with increases in the probability of arrival of informed traders. Differently put, while under the null of an observed price process which impounds all available information and behaves as an Ito semimartingale, EXIT is zero (mixed) normally distributed in the limit, under a natural "frictional" alternative rooted in market microstructure theory, our proposed measure may diverge from zero at a rather fast speed. We study the behavior of EXIT (under the alternative) for an increasing size of transaction costs and an increasing probability of arrival of informed traders. We show significant statistical power in detecting deviations from a world in which securities' prices contain all available information and evolve as implied by frictionless no-arbitrage theories of price formation, i.e., a world in which prices are Ito semimartingales (Duffie, 2008).

EXIT can be computed for every period (every day, say) in our sample. For every period, it would measure the extent of price deviations from an ideal, frictionless world. Under an assumption of time-invariant asymmetric information, EXIT can be viewed as a *liquidity proxy* (the more illiquid an asset, the larger the transaction costs, the more sluggish the price adjustments since informed traders may not be able to profit from their knowledge). In light of our arguments above, it may also be viewed, more broadly, as a *friction proxy*, capturing *jointly* the extent of illiquidity (i.e., leading to larger transaction costs) as well as the extent of asymmetric information. This paper takes the view that the level of asymmetric information (as opposed to the probability of informed trading, which depends on the magnitude of execution costs) is stable or slowly changing. In consequence, we interpret EXIT as a liquidity measure. We will return to the assumption of stable asymmetric information - a classical maintained assumption in the liquidity literature - and will discuss it further in what follows.

Postulating a semimartingale null is natural, in that it is consistent with classical continuous-time modelling in finance. We, however, show that a more internally-consistent (given an accepted "frictional" alternative) null hypothesis is a price formation process in which observed prices have short-memory deviations around the equilibrium Ito semimartingale. These contaminations - justified economically in our framework - are often called, in other literatures, ”microstructure noise”. The addition of microstructure noise under the null does not modify the logic of our test as laid out above. If anything, the addition makes the return process more prone to large deviations. Hence, rejections of the null will provide stronger evidence in favor of the frictional alternative. Assuming a noise-contaminated semimartingale null hypothesis, we show that the asymptotic distribution of EXIT is normal, centered at zero, and with a limiting variance which depends inversely on the variance of the noise, rather than on the infinitesimal variance of the return process as in the no-noise case. The rate of convergence is also faster than in the case without noise.

We demonstrate that EXIT can be employed to test for asymmetric information and, given a level of asymmetric information, to measure the extent of illiquidity. Empirical work on US market returns (proxied here by S&P 500 futures returns) shows that EXIT is larger than zero, and statistically significantly so, for a large number of days corresponding to well-known crisis like the Asia crisis and Lehman's default, thereby providing econometric support for the presence of asymmetric information - and informed trading - over time. The use of EXIT as a liquidity proxy uncovers an economically-meaningful short- and long-run compensation for illiquidity risk in market returns.

A successful empirical literature in finance uses the incidence of daily zero returns (*zeros*) as an illiquidity measure (e.g., Lesmond, 2005, Bekaert et al., 2007, Naes et al., 2011). It is argued that employing price data alone is an important advantage of this approach over existing proxies particularly in markets, like emerging markets, in which information other than that in transaction prices is hard to come by (Bekaert et al., 2007). EXIT shares this useful feature. If setting $\xi_n = 0$, one can indeed interpret the measure that we propose as the proportion of intra-period zero returns over the time interval $[0, T]$. More generally, however, we capture the percentage of price adjustments below a (vanishing, in the limit) threshold. While the applied literature on *zeros* provides empirical motivation for aspects of our approach, we differ from it along a variety of dimensions.

First, differently from the work on *zeros*, we provide a complete theory of inference for the proposed measure under a *null* process cast in the tradition of classical continuous-time finance theory. The alternative process allows for the presence of execution costs (c) and asymmetries in information ($\mathcal{I}$ defines the likelihood of arrival of informed agents). The proposed theory justifies using the measure as an illiquidity proxy (given a certain level of asymmetric information) as well as the construction of two alternative tests: a test for the semimartingale null (under $c = \mathcal{I} = 0$) and a test for a *noise-contaminated* semimartingale null (under $\mathcal{I} = 0$ and $c \neq 0$). Since execution costs are - even in very liquid markets - present, the former test has largely a theoretical value. The latter has, however, both a theoretical and an empirical significance. It allows us to answer the question: do prices behave as semimartingales contaminated by noise, a popular specification in other successful literatures? Equivalently, in our framework it allows us to test for the presence of asymmetric information. We find that information-based trading may lead to sluggish behavior that is, in terms of time-series modelling of the price process, inconsistent with a noise-contaminated semimartingale.

Second, we show that the presence of a threshold, absent in the construction of *zeros*, is both theoretically necessary and empirically important. It is theoretically necessary to develop an inferential theory under a null hypothesis in which the underlying equilibrium process has continuous adjustments and the null distribution is not degenerate at zero. In discrete time, if logarithmic returns were endowed with a continuous density and if $\xi_n = 0$ (as in the literature on *zeros*), EXIT would be identically null for every sample size. The same would occur in continuous time when working with an underlying Ito process, for instance. As said, we operate in continuous time and evaluate the limiting and finite sample properties of the proposed measure (with $\xi_n > 0$). In this context, we discuss the important empirical role played by the sequence ξ_n in trading off size and power. We find that a larger threshold is needed for testing (for the presence of asymmetric information, for example) while a smaller threshold is beneficial when measuring illiquidity through EXIT. In sum, the use of a threshold allows us to exploit the informational content of high-frequency prices for testing as well as for estimation.

Third, guided by the proposed limit theory, we show that solely taking the percentage of "small returns" below a threshold, however small, is sub-optimal in finite samples. Since, even in the absence of illiquidity, there is a likelihood of small returns below the assumed threshold, the expected percentage of these "spurious" small returns has to be subtracted for the measure itself to capture "genuine" small returns induced by illiquidity. EXIT (i.e., the bias-corrected version of IT) accounts for this bias directly.

Fourth, we show that the use of *high-frequency prices* in the calculation of EXIT translates into a considerably less noisy, and therefore more efficient, illiquidity proxy than is the case for *zeros*. This is analogous to improvements (over daily estimates) in volatility estimation obtained by virtue of high-frequency measures of variance, as in the realized variance tradition.

The paper proceeds as follows. Section 2 provides a motivating model of price formation with transaction costs and asymmetric information. We evaluate the properties of EXIT along both dimensions. Section 3 studies the relation between EXIT and execution costs as a function of the sampling frequency and the choice of threshold. We relate EXIT to *zeros* as well as to a suitable high-frequency benchmark. In this context, we provide evidence for its promising performance in correlating with both *nominal* and *effective* costs of transacting. Section 4 discusses an asymptotic theory for EXIT estimates under a classical semimartingale null as well as in a model in which the null allows for microstructure noise contaminations, whose presence is justified structurally in our framework. These results are then contrasted with a frictional alternative implying price sluggishness. In Section 5 we study the finite sample properties of the proposed measure by simulation, and provide information on the choice of the vanishing threshold ξ_n. Section 6 contains empirical work focusing on a test for asymmetric information and the evaluation of short-term and long-term compensations for illiquidity risk in market returns. Useful extensions of the proposed measure, and further discussions, are provided in Section 7. Section 8 concludes. Technical details and issues of implementation are presented in the Appendices.

2 A model of price formation with stale returns

We consider a simple model of price formation featuring transaction costs and traders with different degrees of information. The model captures the effect described in the introduction: if the value of the information signal is larger than the cost of trading, informed traders will act on it and trade. Otherwise, they will choose not to trade, thereby leading to price staleness. For simplicity, the model is written in discrete time. It can, however, be readily viewed as the discretized version of an analogous continuous-time model driven by Brownian shocks.

The model has three components: an equilibrium price process, a midquote adjustment, and observed prices. The equilibrium price process follows (in the logarithm):

$$p_t^e = p_{t-1}^e + \sigma \sqrt{\Delta}\, \epsilon_t, \quad \epsilon_t \sim N(0,1) \tag{2.1}$$

where Δ is the time difference between price adjustments. Hence, equilibrium prices are random walks. The addition of a risk-premium (or a finite variation component, in the language of continuous-time finance) is innocuous. In Section 4, the null hypothesis will, in fact, be stated for a semimartingale price process, i.e., a martingale with drift.

Denote, now, by m_t the (logarithmic) mid-quote of the bid and ask prices at time t. The expected midpoint is assumed to coincide with the expected equilibrium price, i.e., $\mathrm{E}(m_t) = \mathrm{E}(p_t^e)$. In addition, the midpoint adjusts to the equilibrium price with speed given by the parameter δ:

$$m_t = m_{t-1} + \delta(p_t^e - m_{t-1}). \tag{2.2}$$

The larger δ, the faster the adjustment. If $\delta = 1$, $m_t = p_t^e$ at all times.

Finally, the observed price depends on the trader type. Denote by $\mathcal{I}$ the probability of arrival of an informed trader (PAIT), and by $1 - \mathcal{I}$ the probability of arrival of a noise trader. The informed trader knows the equilibrium price and makes his/her decision (buy/sell/do nothing) by comparing the gap between midquote and equilibrium price to the transaction cost c. One can think of c as the half bid-ask spread. In reality, since a large number of transactions occur within the spread, it can be viewed as a more realistic notion of execution cost than the half bid-ask spread. If $|p_t^e - m_t| \leq c$ the informed trader does not trade. If $|p_t^e - m_t| > c$, the observed trading price is

$$p_t = m_t + c 1_{\{p_t^e - m_t > c\}} - c 1_{\{p_t^e - m_t < -c\}}. \tag{2.3}$$

Noise traders just toss a coin, and when they trade the observed price is

$$p_t = m_t + \eta_t c, \tag{2.4}$$

where η_t is a random variable taking the values ± 1 with likelihood 50%. We note that PAIT (the probability of arrival of informed traders) does not coincide with PIN, Easley and O'Hara (1987)'s probability of informed trading. Given the model, in fact, the informed traders may, or may not, trade depending on convenience. While the latter (PIN) is certainly time-varying (given time-varying execution costs), the former (PAIT) is generally assumed to be constant in the literature. As said, we will return to this issue.

The model is a generalization of the data-generating process in Hasbrouck and Ho

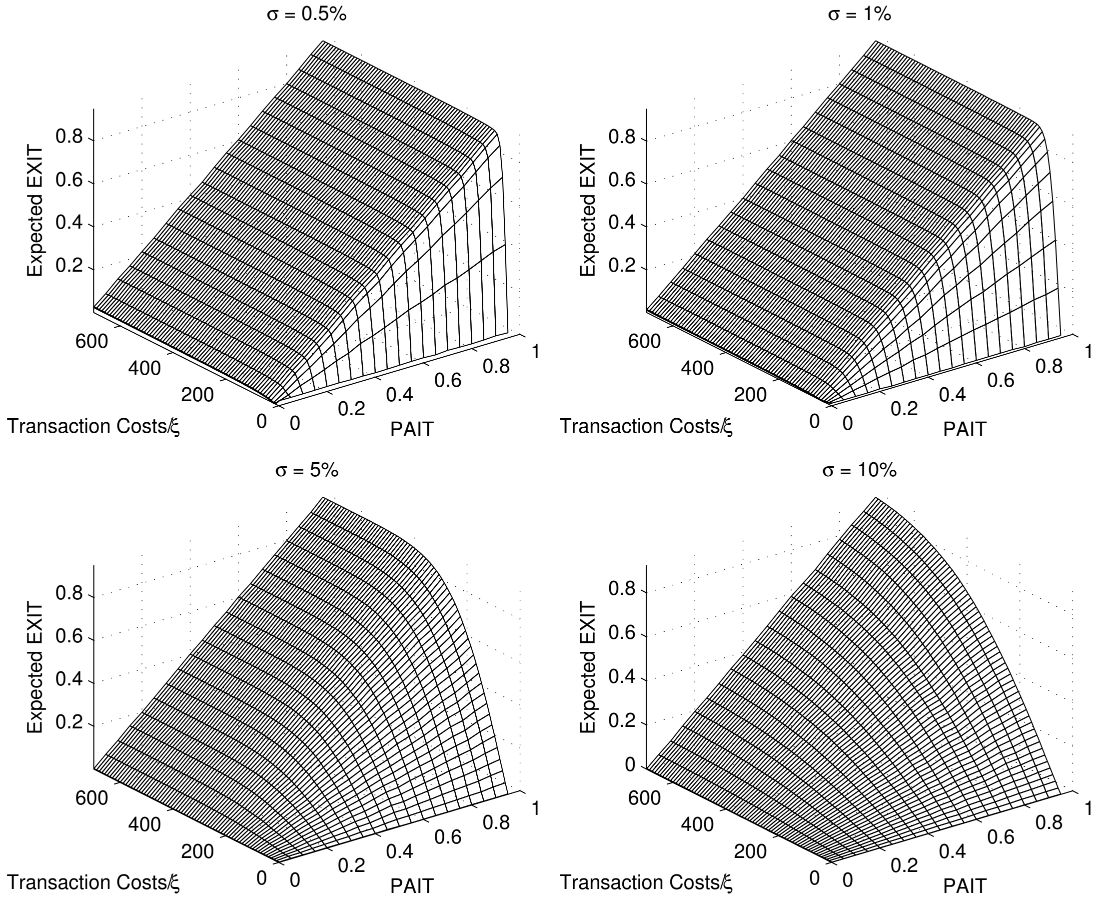

Figure 1: EXIT as a function of transaction cost and PAIT, the probability of arrival of informed traders, computed under the model assumptions in Section 2, with $\delta = 0.46$ and different values of σ.

(1987). The generalization is in Eq. (2.3). Here, we give a role to asymmetric information in leading to sluggishness in the price adjustments. This effect is in agreement with adverse selection models in the tradition of Glosten and Milgrom (1985) and Kyle (1985), justifies formally the arguments (behind the LOT measure and *zeros*) in Lesmond, Ogden and Trzcinka (1999) and Lesmond (2005), and is consistent with the approach in Bandi, Lian and Russell (2011).

They key parameters for our purposes are c (the size of transaction costs) and $\mathcal{I}$ (the probability of arrival of informed traders). In general, given c (or $\mathcal{I}$), an increase in $\mathcal{I}$ (or c) is expected to lead to a larger number of small returns and, hence, higher values of EXIT. By increasing c and $\mathcal{I}$, one can therefore evaluate the statistical power of EXIT under a sensible "frictional" alternative. Next, we do so while setting $\delta = 0.46$ (the solution implied by Hasbrouck and Ho's empirical work) and varying σ, the volatility of the return process.

2.1 EXIT with illiquidity and asymmetric information

Figure 1 shows the mean value of

$$\text{EXIT} = \frac{1}{T}\left(\sum_{i=1}^{n} \Delta_n 1_{\{|p_{t_i}-p_{t_{i-1}}|\leq \xi_n\}} - \sqrt{\frac{2}{\pi}}\frac{\xi_n\sqrt{\Delta_n}}{\sigma}\right) \tag{2.5}$$

for transaction costs in the range $[0 - 0.7]\%$, where Δ_n is the distance (assumed constant, for the time being) between consecutive observations. When the price is missing, we use the last available one. We note that EXIT in Eq. (2.5) is the bias-corrected analogue of IT in Eq. (1.1). This bias correction is conceptually important. Under a frictionless null, there is still a (vanishing, as $n \to \infty$) probability of sluggish behavior. This adjustment (effectively, the limiting expectation of the indicator under the summation sign) is designed to remove this effect and capture genuine staleness. We refer the reader to Section 4 for details.

The value of ξ_n is 10^{-5}, giving a ratio of transaction costs over ξ_n in the range reported in the figures. Each panel corresponds to a different value of volatility, increasing from $\sigma = 0.5\%$ (top-left) to $\sigma = 10\%$ (bottom-right).

For low values of volatility, we observe a one-to-one correspondence between mean EXIT and the probability of arrival of informed traders. This is due to the fact that, if volatility is close to zero, the informed traders never trade since they cannot gain from the $m_t - p_t^e$ gap. For this reason, we observe a percentage of small returns proportional to the probability of arrival of informed traders *independently* of the magnitude of transaction costs. Of course, this is not the case for zero transaction costs. In this case, we observe zero mean EXIT no matter the probability of arrival of informed traders because the observed price essentially coincides with the equilibrium price. Informed traders cannot exploit the $m_t - p_t^e$ gap also in the case when δ approaches 1. For a large speed of adjustment δ, the difference $|m_t - p_t^e|$ is close to zero and, again, we observe a linear relationship between mean EXIT and the probability of arrival of informed traders, no matter whether we are in a low or in a high volatility regime.

For a realistic value of δ, such as the one assumed in the figures, high values of volatility enable the informed traders to take advantage of the $m_t - p_t^e$ gap depending, of course, on the value of transaction costs. In these cases, we observe a non-linear relation between mean EXIT and the couple (c, $\mathcal{I}$). This relation is, again, strongly increasing in both arguments, as expected. Nonetheless, the relation becomes progressively linear with the level of transaction costs and the probability of arrival of informed traders, due to an evident saturation effect.

3 Understanding EXIT

This section illustrates the ability of EXIT to operate as an effective liquidity measure. We compare it to true *nominal* and *effective* transaction costs as well as to suitable benchmarks. We do so by conducting simulations in which the data generating process is a time-varying modification of the model in Section 2. Specifically, in agreement with known empirical regularities, we allow for changing transaction costs across days and, in some cases, auto-correlated order flow, for a *given* level of asymmetric information. We assume that the transaction cost c varies with time according to

$$c_t = c_{t-1} + \sigma_c \epsilon_t,$$

where the ϵ_t are iid normal and $\sigma_c = 10^{-4}$.

3.1 The relation between EXIT, *nominal* execution costs and *zeros*

Liquidity is an elusive concept with various dimensions. Our focus, here, is on execution costs (on "tightness" in the language of Kyle, 1985) rather than on price impacts ("depth" or "resiliency"). Figure 2 shows, for a single trajectory, the path of EXIT compared to the path of transaction costs for three different values of the threshold ξ_n, where ξ_n is expressed in terms of the return volatility σ (see the discussion in section 5.2). EXIT is strongly correlated with transaction costs, with the correlation increasing for smaller values of ξ_n.

When ξ_n is zero, EXIT coincides with the frequency of *intra-daily* zero returns (*zeros*) and, under the Brownian semimartingale assumption, would be zero almost surely. The frequency of daily *zeros* has been employed very successfully in empirical finance work as an illiquidity proxy (e.g., Lesmond, 2005, Bekaert et al., 2007, Naes et al., 2011). We expect the use of intra-daily data, a defining feature of the measure we propose, to lead to estimates that are less noisy and more informative about true transaction costs. To this extent, we present an experiment in which transaction costs vary on a monthly basis. For each month, we compute EXIT using minute-by-minute, hour-by-hour, and day-by-day returns. We use

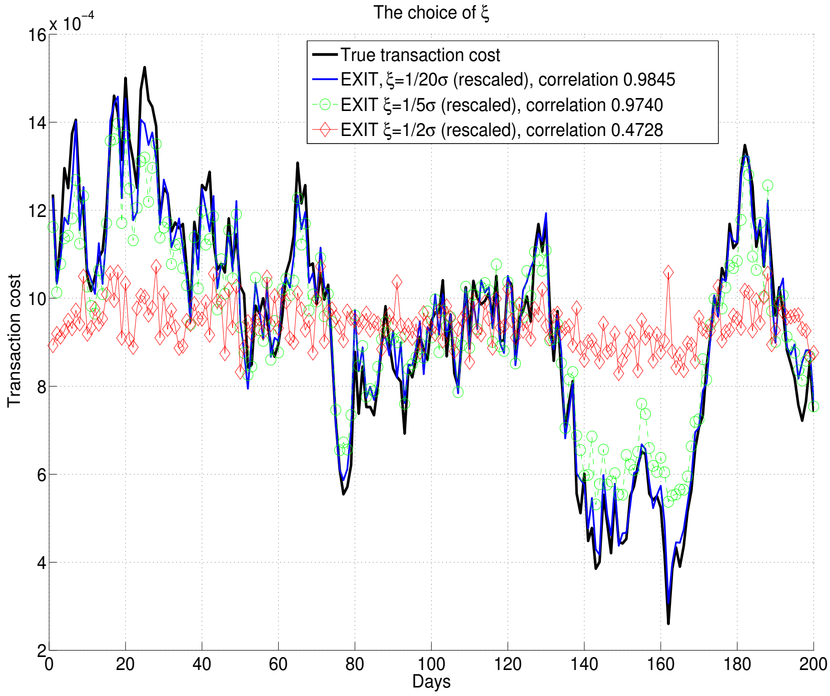

Figure 2: Generated transaction cost and (rescaled) EXIT computed with different values of the bandwidth ξ_n, expressed as a fraction of the volatility σ. The figure also reports the correlation between different measurements of EXIT and transaction costs.

$\xi_n = \sigma/5$, where σ is the return standard deviation. Notwithstanding the use of a threshold, which is irrelevant for the purposes of this comparison, the latter measure based on daily returns is, as emphasized earlier, consistent with recent approaches in the literature focusing on the incidence of zero returns. Figure 3 shows that, while all three measures are correlated with true transaction costs, the correlation rapidly decreases along with the sampling frequency. The benefit of using *high-frequency* information to extract a clearer signal in the measurement of liquidity is, of course, analogous to what is found in the realized variance literature (see, e.g., Andersen and Benzoni, 2009, for a recent review). Just like aggregates of squared intra-daily returns represent more accurate estimates of daily variance than squared daily returns, the frequency of *intra-daily* returns within a vanishing threshold, i.e., EXIT, is found to be a superior illiquidity proxy than straight *daily zeros*.

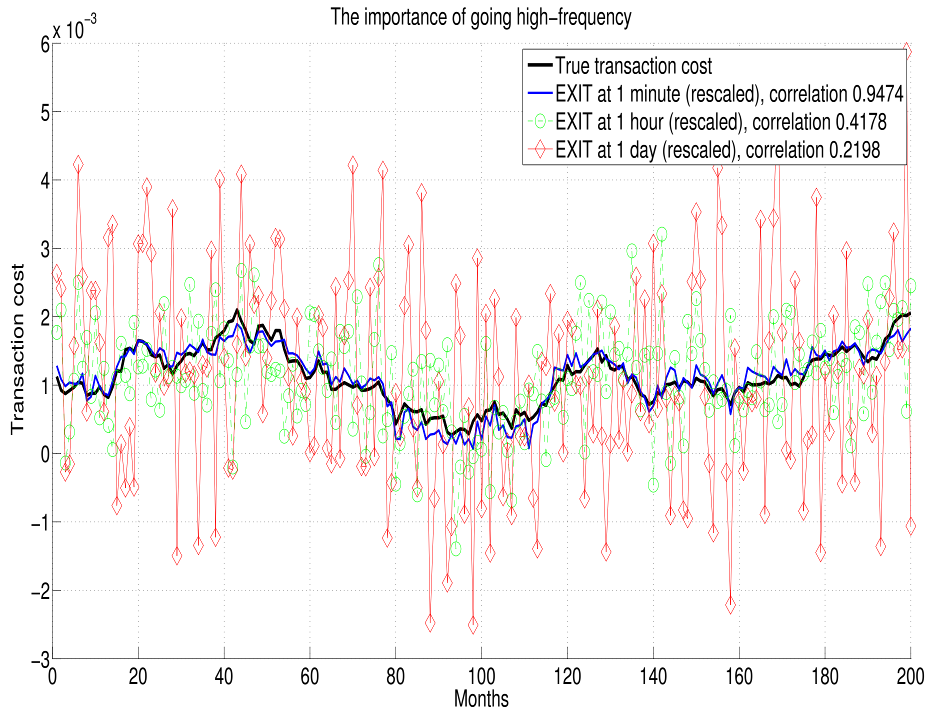

Figure 3: Generated transaction cost and (rescaled) EXIT computed at different frequencies. The figure also reports the correlation between different measurements of EXIT and transaction costs.

3.2 The relation between EXIT, *effective* execution costs, and effective spreads

The constant c may be interpreted as a *nominal* execution cost. Following the intuition in Roll (1984), since a large number of trades occur within the bid/ask spread, c is a more general notion of execution costs than the quoted half bid-ask spread.

An alternative measure of execution costs, for a specific trade indexed by k during the trading day, is the half-spread

$$S_k = Q_k(p_k - p_k^e), \tag{3.1}$$

where p and p^e represent transaction logarithmic price and equilibrium logarithmic price, as in Section 2, and Q_k is an order indicator (+1 for a buy order, −1 for a sell order). The corresponding average over the day, namely

$$\text{True effective (half) spread} = \text{TES} = \frac{1}{n \text{ of trades}} \sum_{k=1}^{n \text{ of trades}} S_k$$

is directly comparable to c and represents an *effective* execution cost. For a noise trader, barring differences between midpoint and equilibrium price, S_k should be similar to c. For informed traders, however, we expect S_k to be negative (i.e., it is a gain, rather than a cost) since information leads to "good" trades, net of transaction costs. In the context of the model in Section 2, the informed traders buy (sell) when $p_k^e > p_k$ ($p_k^e < p_k$). Hence, for these traders, $S_k < 0$ always. In conclusion, once the role of information is taken into account, we expect TES to be smaller than c. In this sense, TES is an "effective" cost.

If the equilibrium price and the nature of the order (buy or sell) where known, TES would, in principle, represent an ideal measure of (average) execution costs. In what follows, we compare EXIT to TES (which is, of course, infeasible) and a suitable, feasible estimate of TES. As illustrated, EXIT presents two fundamental features: it is *solely* based on price information and, even though it could, in principle, be computed using low-frequency data, it employs high-frequency prices for efficiency reasons (see the previous subsection). For a fair comparison, the proposed feasible estimate of TES will also rely on high-frequency transaction prices *only*.

TES estimation poses two well-known issues. Equilibrium prices are always unknown to the econometrician. Similarly, the nature of the orders is only provided by a handful of data sets like Rule 605 and TORQ. This has lead to proxies for both objects. We begin with the first issue.

In the existing microstructure literature, the equilibrium price is often proxied by the current midpoint of the bid/ask spread under the assumption that $E(m) = E(p^e)$, a clear implication of the model in Section 2. Even though midpoint and equilibrium price are the same *in expectation*, midpoints adjust to equilibrium prices due to the learning of the market maker. It is therefore reasonable to believe that $E_k(m_{k+j} - p_k^e) < E_k(m_{k+j-1} - p_k^e)$, thereby yielding lower biases for execution cost estimation the larger j, which is another implication of the model in Section 2. This observation has lead to the use of lagged midquotes m_{k+j} for the computation of S_k, with j varying between 5 minutes (Werner, 2003, among others) and 30 minutes (Bessembinder and Kaufman, 1997, Bessembinder, 1999, and Bacidore and Sofianos, 2002, among others) with other choices, like end-of-the-day midquotes (Werner, 2003), being also possible. The use of a midquote very far out in the future is bound to reduce bias in the estimation of execution costs, but will increase variance. The trade-off between bias and variance can be optimized. More generally, the choice of a future midquote can be based on a suitable criterion, rather than being somewhat ad hoc (Bandi et al., 2011). In all of these cases, the equilibrium price is proxied by the midquote. An alternative procedure, also consistent with learning and price discovery, is to use a future transaction price p_{k+j} rather than a future midquote m_{k+j} to approximate the equilibrium price at the time of the trade (see, e.g., Huang and Stoll, 1996). Since it seems appropriate to construct a benchmark that uses the same information set (i.e., high-frequency transaction prices only) as EXIT, we employ p_{k+j} as in Huang and Stoll (1996), among others.

As for the signing of the trades (Q_k), the typical algorithms are the tick test, the quote test, and the Lee and Ready (1991) method. The tick test (Asquith et al., 2010) classifies a trade as a buy (sell) if the price is higher (lower) than the previous price. If the two prices are the same, the trade is classified based on the previous price. The quote test and the Lee and Ready algorithm (1991) use quotes as well as transaction prices. The Lee and Ready (1991) algorithm, in particular, is thought to be slightly more accurate than the tick test at the cost of using more information (midquotes) than that contained in transaction prices. Using the TORQ data set, Finucane (2000) finds that the Lee-Ready algorithm has a success rate of 84.4% while the tick test classifies correctly 83% of the trades in the sample. Since it only uses transaction prices, we employ the tick method to classify the trades for the purpose of estimating TES.

In sum, the feasible estimate of TES that we implement is:

$$\widehat{\text{TES}} = \frac{1}{n \text{ of trades}} \sum_{k=1}^{n \text{ of trades}} \widehat{S}_k = \frac{1}{n \text{ of trades}} \sum_{k=1}^{n \text{ of trades}} Q_k(p_k - p_{k+j}), \tag{3.2}$$

where $Q_k = 1$ if $p_k > p_{k-1}$ (or $p_k > p_{k-2}$ when $p_k = p_{k-1}$, and so on) and $Q_k = -1$ if $p_k < p_{k-1}$ (or $p_k < p_{k-2}$ when $p_k = p_{k-1}$ and so on). As in Huang and Stoll (1996) and, more recently, Goyenko et al. (2009), we set $j = 5$ and use the price of trade five-minutes after the kth trade. We note that this measure is entirely analogous to one of the three measures used as benchmarks to evaluate the goodness of existing measures of transaction costs by Goyenko et al. (2009), namely their realized spread measure in the Eq. (2). What differentiates $\widehat{\text{TES}}$ from the same measure in Goyenko et al. (2009) is the way in which the order flow is signed.

We assume that the noise traders have a probability p_r to repeat the previous order. In the model in Section 2, we set $p_r = 0.5$, representing uncorrelated order flow. In this subsection, we also consider the case $p_r = 0.9$ yielding order-flow autocorrelation, a well-known stylized fact. As earlier, we use $\xi_n = \sigma/5$. Autocorrelation in order flow is known to be an important factor affecting the performance of execution cost measures relying, like EXIT and $\widehat{\text{TES}}$, only on price observations. Roll's measure, for example, hinges on bid/ask bounce effects. In the absence of these affects, and in the presence of price persistence on the same side of the market, the measure is known to be considerably less effective.

Figure 4, left panels, shows nominal transaction costs (c_t), effective transaction costs (TES_t), EXIT_t and $\widehat{\text{TES}}_t$ over time, for two levels of the probability of arrival of informed traders (0.2 and 0.6). Nominal and effective transaction costs are very highly correlated. As expected, increasing the probability of arrival of informed traders translates into lower levels of TES_t as compared to c_t. In both cases, EXIT_t and $\widehat{\text{TES}}_t$ track execution costs effectively. Increasing order flow autocorrelation (right panels) impacts the performance of EXIT only marginally. It does, however, affect the performance of $\widehat{\text{TES}}_t$ to a large degree. This outcome is due to misclassifications of the nature of the orders caused by clustering in order flow.

This discussion was purposely kept within the theoretical framework of a controlled experiment in which true costs are known and simulated. While these results are suggestive of EXIT's potential, we leave an empirical comparison of EXIT with alternative executions cost measures (which may or may not use high-frequency price information only) for applied future work.

After having discussed the economic significance of EXIT, we now provide a treatment of its statistical properties in continuous time.

4 EXcess Idle Time: asymptotic properties

We begin with conditions on the price process (Assumption 1) and on the sampling process (Assumption 2). The conditions on the price process are classical in continuous-time finance. The conditions on the sampling process allow for unequispaced sampling and involve the quadratic variation of time. We then introduce the probability of flat trading (Assumption 3) as a tool to develop the mathemat-

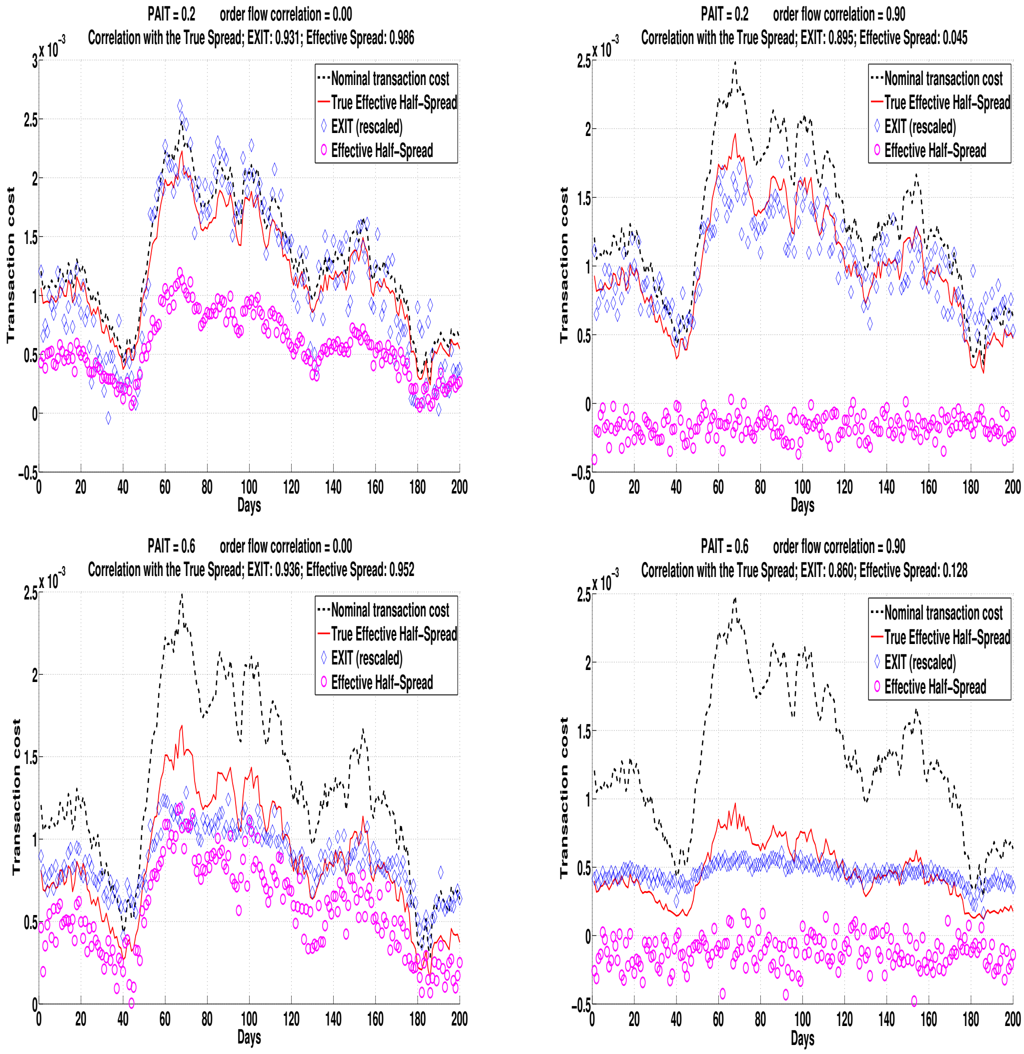

Figure 4: Generated nominal transaction cost, true effective half-spread, rescaled EXIT, and effective half-spread for different values of PAIT (probability of arrival of informed traders) and order flow correlation.

ical properties of idle time under an alternative hypothesis which implies price stickiness. This is designed to capture the economic logic laid out in Section 2.

Assumption 1 (the price process under the null). The real-valued logarithmic efficient price process $\{p_t^e; t \geq 0\}$ is a Brownian semimartingale

$$dp_t^e = \mu_t dt + \sigma_t dW_t,$$

where W_t is a standard Brownian motion, μ_t is a predictable process, and σ_t is cádlág. There exist strictly positive constants C_1, C_2 so that

$$|\sigma_t| + |\mu_t| \;\; \leq \;\; C_1,$$

$$\sigma_t \geq C_2.$$

Further, there exists $\gamma > 0$ and $C_3 > 0$ so that

$$\mathrm{E}[(\mu_s - \mu_t)^2 + (\sigma_s - \sigma_t)^2] \leq C_3(s-t)^\gamma$$

for all $s, t \in [0, T]$ and for $s - t$ small enough.

Assumption 2 (sampling). The process $\{p_t; t \geq 0\}$ is observed at $n+1$ times over $[0, T]$, namely $0 = t_0 < t_1 < ... < t_n = T$. For n sufficiently large, we have

$$\frac{C_4}{n} \leq \min_{1 \leq i \leq n} (t_i - t_{i-1}) \leq \max_{1 \leq i \leq n} (t_i - t_{i-1}) \leq \frac{C_5}{n}$$

for suitable constants C_4 and C_5. Further, for $\beta = \frac{1}{2}, \frac{3}{2}$, and for all $t \in [0, T]$, there exists the limit

$$H_\beta(t) = \lim_{n \to \infty} \left(\frac{n}{T}\right)^{\beta - 1} \sum_{t_i \leq t} (\Delta_{n,i})^\beta,$$

where $\Delta_{n,i} = t_i - t_{i-1}$. Finally, $H_\beta(t)$ is everywhere differentiable with bounded first derivative and, for some $C_6 > 0$ not depending on i,

$$\sup_{t_{i-1} \leq s \leq t_i} \left| H'_\beta(s) - \left(\frac{T}{n}\right)^{1-\beta} (\Delta_{n,i})^{\beta - 1} \right| \leq C_6 n^{-\varepsilon},$$

for some $\varepsilon > 0$.

Assumption 2 contains technical conditions to deal with unequally-spaced observations. When observations are equally-spaced, we have $\Delta_{n,i} = \frac{T}{n}$, $H_\beta(t) = t$ and $H'_\beta = 1$, and Assumption 2 is straightforwardly satisfied. When observations are not equally-spaced, the first condition guarantees that all the $\Delta_{n,i}$ vanish at rate $1/n$, the second that the limit appearing in the asymptotic variance of EXIT exists, and the third that the derivative of H_β converges uniformly over i. When $\beta = 2$, $H_\beta(t)$ has been introduced by Mykland and Zhang (2006) as the quadratic variation of time.

Assumption 3 (the price process under the alternative). The observed price process $\{p_t; t \geq 0\}$ is such that $p_{t_0} = p^e_{t_0}$ and, for $i = 2, \ldots, n$,

$$p_{t_i} = p^e_{t_i}(1 - B_{i,n}) + B_{i,n} p_{t_{i-1}}, \tag{4.1}$$

where $B_{i,n}$ is a triangular array of Bernoulli variates such that

$$\frac{1}{T}\sum_{i=1}^{n}(t_i - t_{i-1})B_{i,n} \xrightarrow{p} p^F, \tag{4.2}$$

where $p^F \in]0,1]$. Moreover, denoting by K_n the supremum of the number of consecutive flat trades, we have,

$$\frac{K_n}{n} \xrightarrow{p} 0. \tag{4.3}$$

Assumption 3 allows for the possibility of no trade and is in line with the behavior of informed traders in the model in Section 2, as described in Eq. (2.3). Informed traders might decide not to trade if it is not convenient for them to do so.

When the Bernoulli variates are iid with probability of no-trading given by p^F, barring unequally-spaced sampling (handled in the proof of Theorem 1), Eq. (4.2) is a classical law of large number for iid sequences and $K_n = O_p(\log n)$ (Schilling, 1990), so that Eq. (4.3) is readily satisfied.

This said, Eqs. (4.2) and (4.3) allow the Bernoulli variates to be correlated with the efficient price, auto-correlated and non identically distributed. In the non identical distribution case, p^F is the (asymptotic) average probability of flat trading. In this more general case, Eq. (4.2) could be replaced by a classical summability condition on the Bernoulli variances and covariances leading to the same law of large numbers. Eq. (4.3) continues to imply that the number of consecutive flat trades diverges at a lower rate than the number of observations. The impact of flat trading on realized volatility measures has been studied by Phillips and Yu (2009).

In what follows, we will denote by $\mathcal{H}_0$ the null hypothesis, described by Assumptions 1 and 2 with $p_t = p_t^e$, and by $\mathcal{H}_1$ the alternative hypothesis in which we add the possibility of flat trading as specified in Assumption 3. We note that $\mathcal{H}_0$ is nested in $\mathcal{H}_1$ and can be obtained by setting $p^F = 0$.

Theorem 1. *Let Assumptions 1, 2, 3 be satisfied.*
(Consistency) As $n \to \infty$, let $\xi_n \to 0$ in such a way that $\xi_n\sqrt{n} \to 0$. Then,

$$\mathrm{IT} = \frac{1}{T}\sum_{i=1}^{n}(t_i - t_{i-1})1_{\{|p_{t_i} - p_{t_{i-1}}| \le \xi_n\}} \xrightarrow{p} \begin{cases} 0 & \textit{under } \mathcal{H}_0 \\ p^F & \textit{under } \mathcal{H}_1 \end{cases}$$

(Stable convergence) As $n \to \infty$, let $\xi_n \to 0$ in such a way that $n^{7/10}\xi_n \to 0$ and $\xi_n n^{3/2} \to \infty$.
Under $\mathcal{H}_0$:

$$\frac{n^{1/4}}{\xi_n^{1/2}}\text{EXIT} = \frac{n^{1/4}}{\xi_n^{1/2}}\frac{1}{T}\sum_{i=1}^{n}\left((t_i - t_{i-1})1_{\{|p_{t_i}-p_{t_{i-1}}|\leq\xi_n\}} - \sqrt{\frac{2}{\pi}}\frac{\xi_n\sqrt{\Delta_i}}{\sigma_{i-1}}\right)$$
$$\stackrel{stably}{\Rightarrow} N\left(0, \sqrt{\frac{2}{\pi}}T^{-\frac{3}{2}}\int_0^T \frac{1}{\sigma_s}H'_{3/2}(s)ds\right).$$

Under $\mathcal{H}_1$:

$$\frac{n^{1/4}}{\xi_n^{1/2}}\text{EXIT} \xrightarrow{p} +\infty.$$

Under the Ito semimartingale null ($p^F = 0$), EXIT converges (stably) to a zero-mean normal distribution whose variability is inversely proportional to the standard deviation of the return process. The rate of convergence is $\left(\frac{\sqrt{n}}{\xi_n}\right)^{1/2}$, where n is the (increasing) number of intradaily, possibly unequispaced, observations and ξ_n is the (vanishing) threshold used in the definition of the estimator. Such a threshold only has to satisfy the condition $\xi_n \sim n^{-\alpha}$ with $\alpha \in (\frac{7}{10}, \frac{3}{2})$ for a central limit theorem to be derived. In essence, updates to an Ito semimartingale price process are too volatile to be contained in a vanishing threshold. Thus, when $p^F = 0$, $1_{\{|p_{t_i}-p_{t_{i-1}}|\leq\xi_n\}} \xrightarrow{p} 0$, as $\xi_n\sqrt{n} \to 0$, and so does EXIT.

When instead $p^F > 0$ (under the alternative), EXIT converges to p^F and, thus, standardized EXIT diverges at the rate $\frac{n^{1/4}}{\xi_n^{1/2}}$.

Remark 1 (Jumps). Addition of Poisson jumps to the Ito semimartingale in Assumption 1 would not change the convergence in probability to zero of idle time under the null. This can be easily shown using results in Mancini (2009). Since the bandwidth employed in Theorem 1 has to satisfy $\xi_n\sqrt{n} \to 0$ for consistency, these jumps will always be above the bandwidth asymptotically. In this respect, idle time is robust to jumps. Empirically, the presence of jumps would decrease the value of EXIT.

Remark 2 (Testing the semimartingale hypothesis). If one were to consider the model in Section 2 and set $c = \mathcal{I} = 0$ (with $\delta = 1$), then $p_t = p_t^e$ and *observed* prices would be semimartingales. Thus, the asymptotic theory in Theorem 1 could, in principle, be used to test the semimartingale null against a price formation model in which $c \neq 0$ and $\mathcal{I} \neq 0$ *jointly*, that is in which $p^F > 0$.

Since, economically and in the data, $c > 0$, however small, this test has solely a theoretical relevance. The next subsection discusses an analogous test with broad empirical implications.

4.1 Asymptotics with noise

The addition of market microstructure noise contaminations is consistent with the economic model in Section 2 (see Remark 5 below for further discussions). This addition only makes the logic of our assumed approach more compelling. A noise-contaminated Ito martingale price process moves outside of a vanishing threshold more frequently than in the no-noise case, thereby leading to a faster rate of convergence to zero. Assumption $1'$ introduces the assumed *contaminated* process. Theorem 2 gives the limiting distribution of EXIT in the noise case under the null (Assumption $1'$) and the alternative (Assumption $3'$).

Assumption $1'$ (the price process under the null with microstructure noise). Let the equilibrium logarithmic price process be distorted as $\widetilde{p}_{t_i} = p^e_{t_i} + \eta_{t_i}$, where η_{t_i} is IID in discrete time, independent of p^e_t, and such that $\eta \sim N(0, \sigma^2_\eta)$.

Assumption $3'$ (the price process under the alternative with microstructure noise). Assumption 3 continues to hold with $p_{t_0} = \widetilde{p}_{t_0}$ and Eq. (4.1) replaced by

$$p_{t_i} = \widetilde{p}_{t_i}(1 - B_{i,n}) + B_{i,n} p_{t_{i-1}}. \tag{4.4}$$

Theorem 2. *Let Assumptions $1'$, 2, $3'$ be satisfied.*
(Consistency) As $n \to \infty$, let $\xi_n \to 0$. Then,

$$\text{IT} = \frac{1}{T}\sum_{i=1}^{n}(t_i - t_{i-1})1_{\{|\widetilde{p}_{t_i} - \widetilde{p}_{t_{i-1}}| \le \xi_n\}} \xrightarrow{p} \begin{cases} 0 & \textit{under } \mathcal{H}_0 \\ p^F & \textit{under } \mathcal{H}_1 \end{cases}$$

(Weak convergence) As $n \to \infty$, let $\xi_n \to 0$ in such a way that $n\xi_n^5 \to 0$ and $\xi_n n \to \infty$.
Under $\mathcal{H}_0$:

$$\begin{aligned} \frac{n^{1/2}}{\xi_n^{1/2}}\text{EXIT} &= \frac{n^{1/2}}{\xi_n^{1/2}}\frac{1}{T}\sum_{i=1}^{n}\left((t_i - t_{i-1})1_{\{|\widetilde{p}_{t_i} - \widetilde{p}_{t_{i-1}}| \le \xi_n\}} - \sqrt{\frac{2}{\pi}}\frac{\xi_n \Delta_i}{\sigma_\varepsilon}\right) \\ &\Rightarrow N\left(0, \sqrt{\frac{2}{\pi}}T^{-1}\frac{1}{\sigma_\varepsilon}H_2(T)\right), \end{aligned} \tag{4.5}$$

where $\sigma_\varepsilon = \sqrt{2\sigma_\eta}$.
Under $\mathcal{H}_1$*:*

$$\frac{n^{1/2}}{\xi_n^{1/2}}\text{EXIT} \xrightarrow{p} +\infty.$$

Under the noise-contaminated Ito semimartingale null ($p^F = 0$), EXIT converges to a zero-mean normal distribution whose variability is now inversely proportional to the standard deviation of the dominating component in the return process, i.e., microstructure noise. The bias vanishes at a faster rate than in the no-noise case (ξ_n/n versus $\xi_n/\sqrt{n}$). The rate of convergence to the limiting normal distribution is, similarly, faster ($\frac{n^{1/2}}{\xi_n^{1/2}}$ versus $\frac{n^{1/4}}{\xi_n^{1/2}}$).

Remark 3. (Endogenous noise) Classical forms of endogeneity (correlation between the noise and the equilibrium price process) can be introduced without effecting the outcome of Theorem 2. Consider, for example, the specification in Nolte and Voev (2012) where

$$\widetilde{p}_{t_i} = p_{t_{i-1}} + \phi r_{t_i} + \eta_{t_i} \tag{4.6}$$

with $r_{t_i} = p_{t_i} - p_{t_{i-1}}$. Then, of course,

$$\widetilde{p}_{t_i} - \widetilde{p}_{t_{i-1}} = p_{t_i} - p_{t_{i-1}} + \varepsilon_{t_i},$$

where $\varepsilon_{t_i} = (\phi - 1)(r_{t_i} - r_{t_{i-1}}) + \eta_{t_i} - \eta_{t_{i-1}}$. If $\phi = 1$, we recover Assumption 3. If $\phi < 1$, the covariance between $p_{t_i} - p_{t_{i-1}}$ and ε_{t_i} is negative, an empirical possibility and a theoretical finding discussed by many authors. This model can also be viewed as a more general specification of that in Kalnina and Linton (2008), where r_{t_i} in Eq. (4.6) is replaced by the difference of the driving Brownian motions ($W_{t_i} - W_{t_{i-1}}$). In both cases, as we show in proof of the remark, the limiting distribution in Theorem 2 holds. Hence, the derived limiting distribution is robust to current specifications for endogeneity in the noise.

Remark 4 (Dependent noise). Dependence in order flow and, as a result, dependence in market microstructure noise can be easily accommodated. Here, we employ an AR(1) model for the price contaminations. A more general ARMA structure can be assumed along identical lines. If $\eta_i = \rho\eta_{i-1} + u_i$, then

$$\begin{aligned}\frac{n^{1/2}}{\xi_n^{1/2}}\text{EXIT} &= \frac{n^{1/2}}{\xi_n^{1/2}}\frac{1}{T}\sum_{i=1}^{n}\left((t_i - t_{i-1})1_{\{|\widetilde{p}_{t_i}-\widetilde{p}_{t_{i-1}}|\leq\xi_n\}} - \frac{1}{\sqrt{\pi}}\frac{\sqrt{1-\rho^2}}{\sqrt{1-\frac{1}{2}\rho^2}}\frac{\xi_n\Delta_i}{\sigma_u}\right) \\ &\Rightarrow N\left(0, \frac{1}{\sqrt{\pi}}\frac{\sqrt{1-\rho^2}}{\sqrt{1-\frac{1}{2}\rho^2}}T^{-1}\frac{1}{\sigma_u}H_2(T)\right)\end{aligned}$$

under the same conditions as in Theorem 2, when $p^F = 0$.

Remark 5 (On the importance of noise contaminations under the null). One can derive a natural null hypothesis by setting to 0 the probability of arrival of informed traders in the "frictional" alternative in Section 2, see Eq. (2.4). This would give rise to short-memory deviations (due to the adjustments of the midpoint as well as noise trading) around the underlying semimartingale and, hence, short-range dependent market microstructure noise. Said differently, an internally-consistent (with the alternative in Section 2) null hypothesis should - in agreement with the approach in this subsection - contain short-memory noise.

Remark 6 (Testing the noise-contaminated semimartingale hypothesis). As shown formally in Theorem 2, the presence of noise leads, in our framework, to a more powerful test. In addition, the distribution in Theorem 2 can be employed to test a noise-contaminated semimartingale null against a frictional alternative induced by asymmetric information. Equivalently, it can be used to test for absence of asymmetric information ($\mathcal{I} = 0$ and $c \neq 0$, leading to $p^F = 0$) against presence of asymmetric information ($\mathcal{I} \neq 0$ and $c \neq 0$, leading to $p^F > 0$) under the maintained assumption of presence of execution costs, something which is empirically warranted. We do so in Section 6.

5 EXcess Idle Time: finite sample properties

This section evaluates finite sample performance of the proposed measure and corresponding tests. We accommodate stochastic volatility, intraday effects, microstructure noise and rounding of the simulated prices. We study how the behavior of EXIT depends on the choice of the threshold ξ_n under the noise-contaminated semimartingale null. Importantly, we also provide practical guidance on the implementation of EXIT on data and, in particular, on the choice of the threshold ξ_n.

5.1 Monte Carlo experiments

We simulate a one-factor diffusion model with stochastic volatility. The model is described by the pair of stochastic differential equations

$$\begin{array}{lcl} dp_t^e & = & \mu\, dt + \gamma_{t,\tau} c_\sigma \sigma_t dW_{p,t} \\ d\log\sigma_t^2 & = & (\alpha - \beta \log\sigma_t^2)\, dt + \eta dW_{\sigma,t}, \end{array} \tag{5.1}$$

where W_p and W_σ are standard Brownian motions with $\mathrm{corr}\,(dW_p, dW_\sigma) = \rho$ and σ_t is a stochastic volatility factor. We use the model parameters estimated by Andersen et al. (2002) on S&P500 prices: $\mu = 0.0304$, $\alpha = -0.012$, $\beta = 0.0145$, $\eta = 0.1153$, $\rho = -0.6127$, where the parameters are expressed in daily units and returns are in percentage. We further set $c_\sigma = 2$, which calibrates the daily volatility to nearly 20% in annual terms. In addition, we add a multiplicative intraday effect

$$\gamma_{t,\tau} = \frac{1}{0.1033}(0.1271\tau^2 - 0.1260\tau + 0.1239),$$

where τ is the fraction of a day elapsed from opening ($\tau = 0$ at the beginning of the day and $\tau = 1$ at the end of the day), and the parameters in $\gamma_{t,\tau}$ have been calibrated on S&P500 intraday returns with the constraint $\int_0^1 \gamma_{t,\tau} d\tau = 1$. The numerical integration of the system (5.1) is performed with the Euler scheme, using a discretization step of $\Delta = 1$ second. Each day, we simulate $7 \times 60 \times 60$ steps, that is we simulate second-by-second data for seven hours. At every transaction price, we add a microstructure noise shock to every transaction price leading to

$$\tilde{p}_t = p_t^e + \eta_t$$

with η_t IID normally distributed with zero mean and variance σ_η^2. We set $\sigma_\eta^2 = c_\sigma^2 e^{\alpha/\beta}/(7 \times 60 \times 60) = 6.94 \times 10^{-5}$ so that, at the frequency of one second, the ratio of the average $\sigma_t^2\Delta$ with the microstructure noise variance (the signal-to-noise ratio) is equal to one.

We compute EXIT using *one-minute* returns, that is with $n = 420$. The implementation requires a preliminary estimate of spot volatility and of the variance of market microstructure noise. We detail estimation of both quantities in Appendix B. To verify the size properties of the test, we focus on the statistics

$$z = \frac{\mathrm{EXIT}}{\sqrt{V_{\mathrm{EXIT}}}},$$

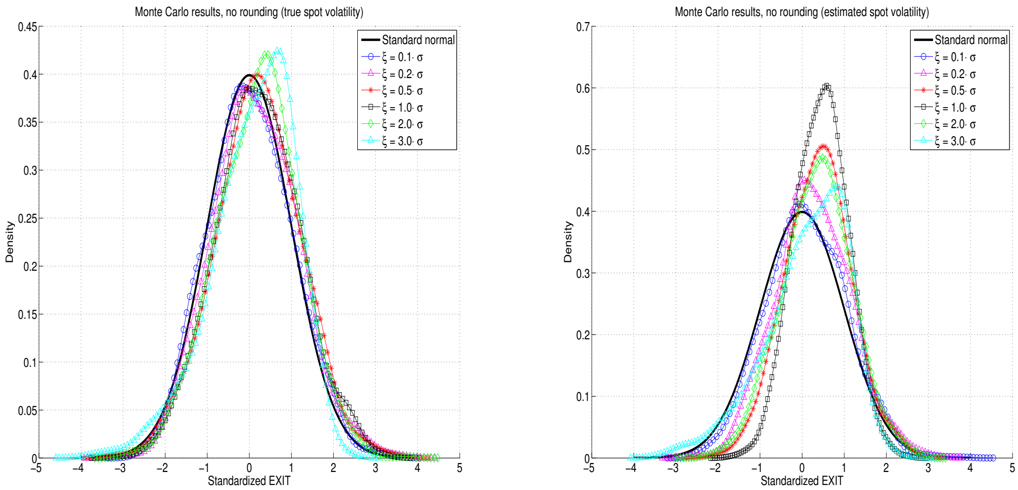

Figure 5: Monte Carlo results: the case with no rounding. Left panel: we use the generated (infeasible) spot volatility. Right panel: we estimate spot volatility with Eq. (B.1).

as given by Eq. (B.2) and (B.3), which, asymptotically, is standard normal (c.f. Theorem 2). We generate $1,000$ replications.

In the first set of simulations, prices are not rounded. In this case, the correct scale of ξ_n is the price volatility and, for this reason, Figure 5 reports the different values of ξ_n not in absolute terms but relative to the average volatility of one-minute returns, denoted by σ. The left panel refers to the infeasible case in which we use the true unobservable volatility. The right panel refers to the feasible case in which we estimate minute-by-minute volatility. We see agreement with the asymptotic standard normal for values of ξ_n up to three times the local standard deviation. Size distortions appear due to noise in the estimates of the spot volatility but they are small for all of the considered threshold values.

In the second set of simulations, we round prices to \$0.01 (one penny), in order to mimic the actual behavior of US stocks. Rounding may be an important source of *finite sample* size distortions, especially for low prices. Indeed, rounding affects prices instead of the logarithmic prices involved in the estimation of EXIT, making it more "aggressive" on returns when the price is relatively lower. For this reason, we consider two starting points for our simulations: $P_0 = 5$ and $P_0 = 50$. We also report different values of ξ_n not only relative to σ, but also relative to $min - tick$, which is defined as

$$min - tick = \frac{(\Delta P)_{\mathsf{MIN}}}{< P >},$$

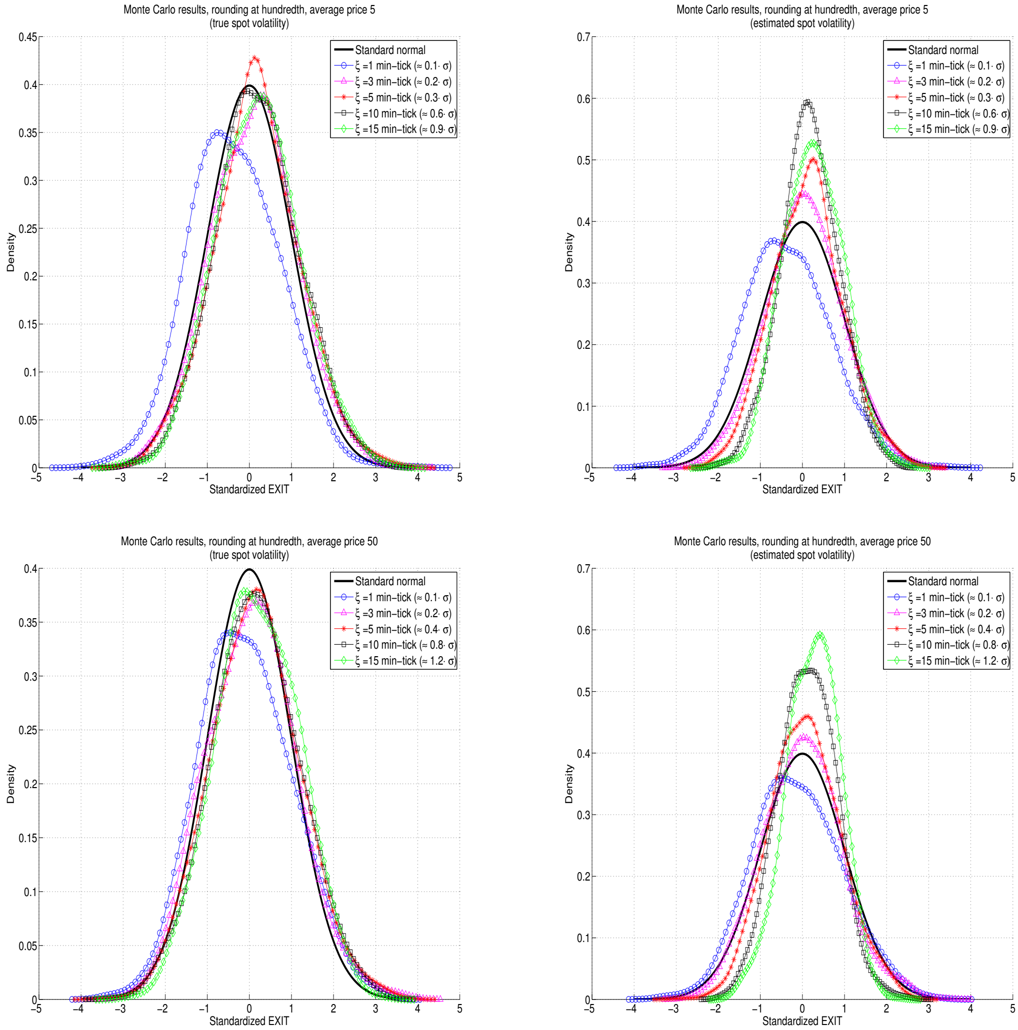

Figure 6: Monte Carlo results: the case with rounding. Left panels: we use the generated (infeasible) spot volatility. Right panels: we estimate spot volatility with Eq. (B.1).

where $(\Delta P)_{\mathsf{MIN}} = 0.01$ is the price discreteness and $< P >$ is the average price in the daily replications. The quantity $min - tick$ is effectively the "minimum" logarithmic price difference. Figure 6 shows the results. As before, the panels on the left refer to the infeasible case in which we use the true unobservable volatility. The panels on the right refer to the feasible case in which we estimate minute-by-minute volatility. The top panels refer to the case $P_0 = 5$; the bottom panels refer to the case $P_0 = 50$. If ξ_n is too low, rounding induces an unnecessary size distortion. Such a distortion disappears if ξ_n is large enough with respect to the assumed price discreteness.

Summarizing, our Monte Carlo study shows that, in a realistic market setting, EXIT's asymptotic distribution has satisfactory small-sample size properties if ξ_n is big enough with respect to the price discreteness induced by rounding. We also show that the return volatility has a negligible impact on size.

5.2 Threshold selection

We now discuss the choice of the bandwidth ξ_n. Assume we observe n equally spaced intraday returns, and denote by σ_d the daily return volatility. It is convenient to express (as we do in Section 3) ξ_n in terms of the return volatility, that is,

$$\xi_n = \alpha \cdot \frac{\sigma_d}{\sqrt{n}} = \alpha \cdot \sigma, \tag{5.2}$$

where $\sigma = \sigma_d/\sqrt{n}$ is the average volatility of intradaily returns. The simulated experiments in Section 3 suggest that equation (5.2) should be used with a small value of α (see Figure 2) to maximize the power of EXIT in *measuring* liquidity, that is to maximize the correlation between EXIT and transaction costs. Coherently, setting a small α is what we do in Section 6 to investigate the presence of a liquidity premium in market returns.

However, if we are interested in *testing*, correct sizing is important. The Monte Carlo experiments in Subsection 5.1 show that, in this case, a small value of α is not advisable. When testing, as in Subsection 6.1, we use Eq. (5.2) with $\alpha = 1$.

Notice, however, that when n is too large, setting $\alpha = 1$ may not translate into a large enough threshold, i.e., a threshold which satisfies:

$$\xi_n = n_\xi \cdot \frac{(\Delta P)_{\mathsf{MIN}}}{< P >}, \tag{5.3}$$

where $(\Delta P)_{\mathsf{MIN}}$ is the price discreteness (\$0.01 for US stocks), $< P >$ is the average daily price, and n_ξ at least larger than 1. In order to satisfy both Eq. (5.2), with $\alpha = 1$, and Eq. (5.3) we derive ξ_n from Eq. (5.3) and suitably select the sampling frequency so that $n = \sigma_d^2/\xi_n^2$. In essence, since $\sigma_d/\sqrt{n}$ might not deliver - when multiplied by $\alpha = 1$ - a large enough threshold, one may not be able to use the full sample of high-frequency observations. Given ξ_n set as a function of price discreteness, for *testing* we recommend choosing n so as to guarantee $\alpha = 1$. This procedure is used in the empirical work, to which we now turn.

6 Staleness in market returns

This section tests for asymmetric information at the market level. Given a likelihood of informed-based trading, we evaluate short-term and long-term compensations for illiquidity risk in market returns through EXIT. We use S&P 500 futures prices from April 28, 1982 to February 5, 2009, for a total of 6,669 days.

6.1 Testing for asymmetric information

Under the frictional alternative in Section 2, a necessary condition for EXIT to be asymptotically large is the presence of asymmetries in information leading to lack of trading (for large enough execution costs). Absent informed traders, EXIT is expected to be small irrespective of the magnitude of execution costs (unless large transaction costs induce lack of trading on the part of the noise traders, an empirical possibility which - albeit formally outside of the model in Section 2 - provides further economic justification for using EXIT as an illiquidity proxy). This observation offers a natural conceptual framework to test - using the limiting results in Section 4 and the finite sample considerations in Section 5 - for the presence of asymmetric information in the determination of market prices.

To this extent, we compute EXIT for every day in the sample as described in Appendix B. The threshold ξ_n is selected using Eq. (5.3) with $n_\xi = 5$ and a sampling frequency n chosen to fulfill Eq. (5.2) with $\alpha = 1$. To estimate σ_d, we use the square root of the threshold bipower variation of Corsi et al. (2010). Thus, prices are equispaced. If a price is missing, we use the previous tick.

Figure 7 provides a graphical representation of the outcome of the test (implemented for each day in the sample). We plot daily EXIT standardized by its standard error, derived in Theorem 2 and estimated in Appendix B, over time. We also plot (with dotted lines) 95% confidence bands to facilitate the analysis.

T-ratios constructed using EXIT spike in correspondence with well-known crisis (labelled in the figure). We do not interpret these results as being symptomatic of time-varying asymmetric information (although, as discussed in Section 7, this possibility cannot be excluded). Rather, we emphasize that the proposed test has more power in times of crisis corresponding to low values of EXIT's variance. Asymptotic considerations (made explicit in Theorem 2) imply that this variance is an inverse function of the noise variance σ_ε^2 (see Eq. 4.5). Finite sample considerations (derived from the proof of Theorem 2 and accounted for in Appendix B

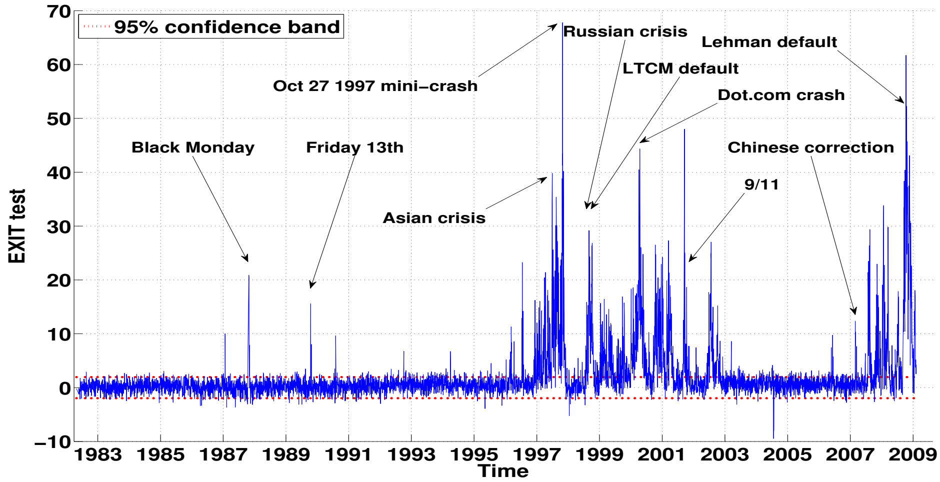

Figure 7: Daily time series of EXIT's asymptotic t-ratios.

for the sake of superior finite sample performance) indicate that this variance is also an inverse function of the variance of the underlying equilibrium price σ^2 (see Eq. B.3). In addition, as shown in other literatures, σ_ε^2 and σ^2 are correlated.

In essence, should EXIT remain large, thereby implying sluggishness in the price adjustments, this effect would be - as compared to tranquil times of low volatility and low noise - *relatively* more indicative of information-induced lack of trading. High volatility/high noise events like the Asian crisis, the dot-com crash and, more recently, Lehman's default are, therefore, bound to be more informative about the presence of asymmetric information than more normal times. Given econometric support for information-based trading, we now turn to illiquidity.

6.2 Illiquidity premia

In Section 3 we argued that EXIT may be used effectively (for any *given* level of asymmetric information) as a liquidity proxy, one which is more robust to clustering in order flow than suitable benchmarks. We now ask the question: does the use of EXIT as a liquidity proxy (under an assumption of slowly time-varying asymmetric information) lead to an illiquidity premium in market returns? More broadly, should asymmetric information be highly volatile, does the use of EXIT as a friction proxy lead to a risk compensation? We approach this issue in two ways. First, we focus on short-term compensation for illiquidity risk. The logic of the procedure is in the spirit of Amihud (2002). We model the time series evolution of EXIT, forecast it over a specific (short-term) horizon, and regress

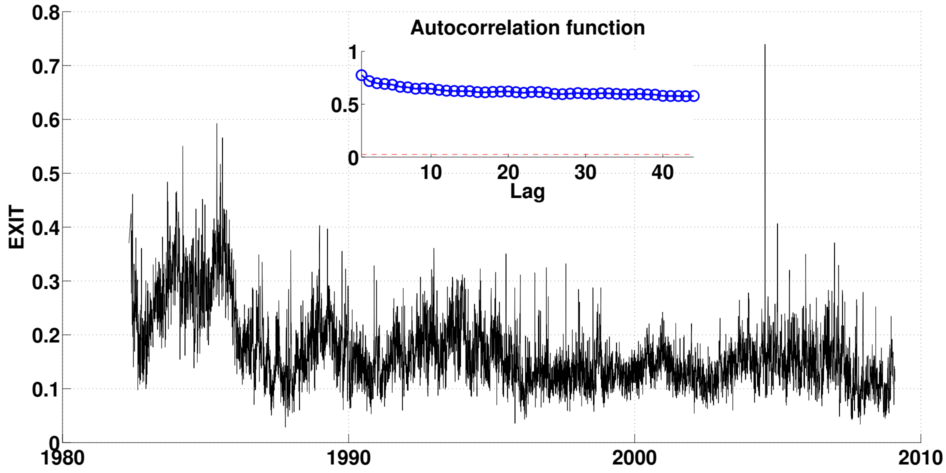

Figure 8: Daily time series of EXIT on S&P500 futures. In the inset, we report the autocorrelation function for the first two months.

excess market returns on the resulting forecasts. If a risk compensation exists, the resulting slope estimates should be positive and statistically significant. Our findings are consistent with this logic. Next, we turn to low-frequency dynamics and employ the backward/forward regressions recently introduced by Bandi and Perron (2008) and justified formally by Bandi et al. (2012). This procedure views economic time series as a sum of components or *details* with different levels of periodicity and persistence. Very little structure is imposed on the short-run dynamics and, in particular, no assumptions are made on the transition between high- and low-frequency time-series evolution, something which would be required by the previous, more traditional, approach. Forward aggregation of excess market returns and backward aggregation of the liquidity proxy provides a way to identify the layer (or frequency) in the cascade of shocks affecting the economy over which an illiquidity compensation may exist. We show that there might be low-frequency components of EXIT with decade-long cycles which are autoregressive and have predictive power for low-frequency components of excess market returns, i.e., a low-frequency compensation for illiquidity risk. These components are hidden by short-term noise. The use of two-way aggregation is effective in extracting their signal.

Because - in this section - we aim to maximize the statistical power of our proposed measure, we compute daily time series of EXIT from intraday one-minute returns using Eq. (5.2) with $\alpha = 1/20$. The use of $\alpha = 1/5$ leaves all of the following results unchanged.

6.2.1 The short run

The time series evolution of EXIT is displayed in Figure 8, along with the autocorrelation function. The memory of the process is considerable. To reproduce this feature, we employ an heterogeneous autoregressive specification (HAR) as in Corsi (2009). Denote by EXIT_t the value of EXIT on day t and write

$$\text{EXIT}_{t:t+h-1} = \alpha_h + \beta_h^{(d)}\text{EXIT}_{t-1:t-1} + \beta_h^{(w)}\text{EXIT}_{t-5:t-1} + \beta_h^{(m)}\text{EXIT}_{t-22:t-1} + \varepsilon_{t+h-1}, \tag{6.1}$$

where the ε's are forecast errors and

$$\text{EXIT}_{t_1:t_2} = \frac{1}{t_2 - t_1 + 1} \sum_{t=t_1}^{t_2} \text{EXIT}_t$$

is a generic average of EXIT values (between days t_2 and t_1, included). Hence, the parameters $\beta_h^{(d)}$, $\beta_h^{(w)}$ and $\beta_h^{(m)}$ correspond to daily, weekly, and monthly averages (used here as predictors) over a forecasting horizon h. The horizon h varies between one day ($h = 1$) and one month ($h = 22$). The dependence between the daily, weekly, and monthly parameters and the forecast horizon is represented in Figure 9. All estimated coefficients are very significant (for all h values between one day and one month) with R^2 values comfortably high.

HAR specifications have been used successfully to model variance dynamics. To this extent, in order to facilitate interpretation, the graphs in Figure 9 provide analogous representations for the parameters of an identical HAR model for logarithmic variance.[1] As in the case of variance, the relative magnitude of the individual parameters (and, in consequence, the relative impact of individual regressors) tends to reach a peak for forecasts conducted over analogous horizons. For example, the weekly averages have more of an impact on prediction for h in a neighborhood of 5 than for h close to a month. The larger monthly coefficients relative to the daily and weekly coefficients (for all h) in the case of EXIT point to the superior memory of EXIT as compared to logarithmic variance.

Denote, now, by $R_{t:t+h}$ the excess return, with respect to the three-month T-bill rate, of the S&P 500 futures returns from t to $t + h$. We estimate the simple specification

$$R_{t:t+h} = a_h + b_h\widetilde{\text{EXIT}}_{t+1:t+h} + \xi_{t+h}, \tag{6.2}$$

[1] We use realized variance estimates constructed using 5-minutes returns.

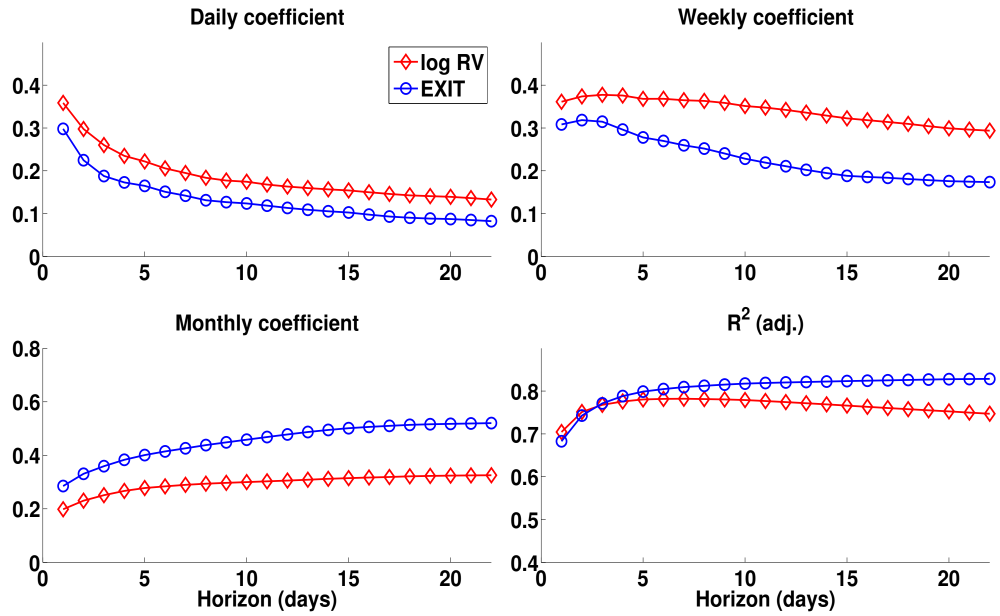

Figure 9: Estimates and adjusted R^2 of the HAR model applied to $\log RV$ and EXIT (with $\xi = \frac{1}{20}\sigma$), respectively.

where the ξ's are forecast errors and $\widetilde{\text{EXIT}}_{t+1:t+h}$ is an EXIT forecast obtained using the model in Eq. (6.1). To run the regressions, we use least-squares with overlapping observations and Newey-West standard errors with $2(h-1)$ lags. The estimated slope coefficients $\widehat{b}_h$ are displayed in Figure 10 for different forecasting horizons (the red dashed line is the upper bound of a two-sided confidence band under the null of no relation). The estimates reveal a positive, albeit statistically mildly significant, compensation for illiquidity risk. The estimated slopes are increasing, and become relatively more significant, with the horizon.

In order to evaluate robustness to a potential variance risk premium, we estimate the specification:

$$R_{t:t+h} = a_h + b_h \widetilde{\text{EXIT}}_{t+1:t+h} + c_h \widetilde{RV}_{t+1,t+h} + \xi_{t+h}, \tag{6.3}$$

where $\widetilde{RV}_{t+1,t+h}$ is a variance forecast obtained from Eq. (6.1) applied to logarithmic variance (the logarithmic forecasts are then exponentiated). The estimated slope coefficients $\widehat{b}_h$ and $\widehat{c}_h$ are reported, again for different h values, in Figure 11 (top and bottom panel, respectively). While a variance risk premium cannot be detected in this sample, a mildly significant short-term illiquidity risk premium is confirmed.

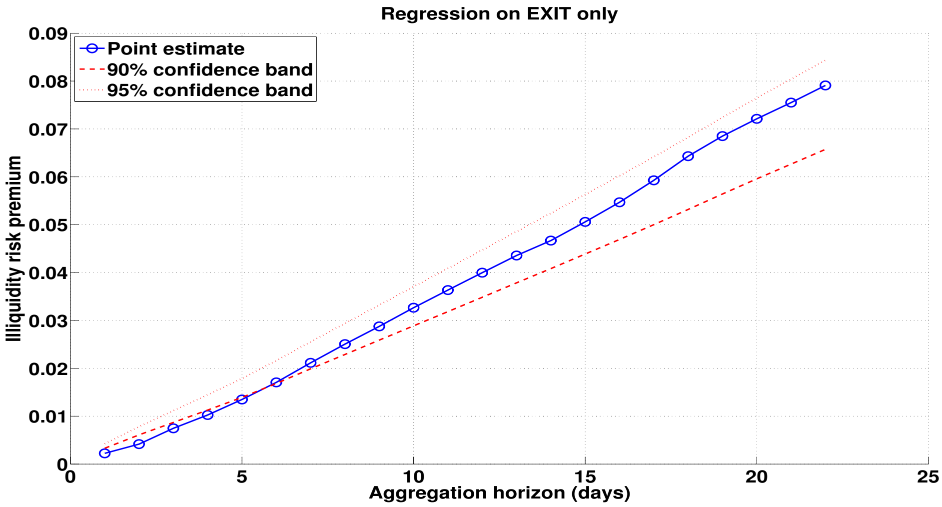

Figure 10: Illiquidity premium for S&P 500 futures returns, measured with Eqs. (6.2)-(6.1).

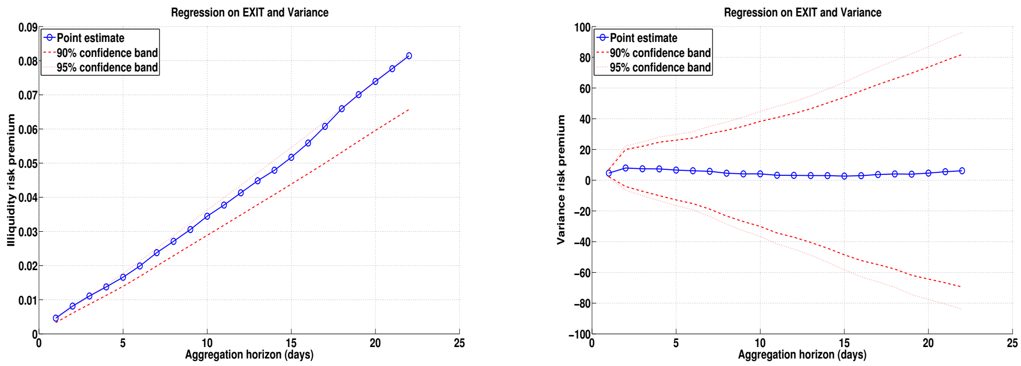

Figure 11: Illiquidity (left panel) and variance (right panel) premium for S&P 500 futures returns, measured with Eqs. (6.3) and (6.1) applied both to EXIT and to logarithmic variance.

The existence of short-term compensations for variance risk is well-known to be elusive. Long-run variance risk premia have, however, been detected both with market variance (Bandi and Perron, 2008) and with consumption variance (Tamoni, 2011). In what follows, we turn to low frequencies and use a low-pass filter (the two-way aggregation method suggested by Bandi and Perron, 2008) to provide evidence of a similar illiquidity risk premium in long-run market returns. We do so without making strong assumptions on short-run dynamics (for EXIT and market returns) and, in particular, without either assuming that suitable long-run forecasts can be implied from a specific short-run model or modeling low-frequency dynamics by virtue of assumed low-frequency specifications.

6.2.2 The long run

Consider the model

$$R_{t:t+h} = a_h + b_h \text{EXIT}_{t-h+1:t} + \xi_{t+h}$$

with h ranging from 252 (one year) to 8×252 (eight years). Future returns over h are regressed onto backward aggregates of EXIT. The theoretical justification for two-way aggregation has been spelled out in Bandi et al. (2012) using a multi-resolution view of observed time series. If slow-moving components of the illiquidity proxy predict themselves, as well as slow-moving components of the price process, backward-forward aggregation will work as an effective signal extraction mechanism. In particular, the significance of the slope estimates will reach a peak in correspondence with the periodicity of the (illiquidity and market return) components (or *details*) which are connected by a predictive relation. Said differently, we view the long-run predictive relation between market returns and illiquidity as being a property of components which may be hidden by short-term contaminations. Two-way aggregation is effective in reducing these (possibly unrelated) short-term contaminations while revealing (i) whether slow-moving components are linked by a relation at all and (ii) what their periodicity is.

The slope estimates have a clear increasing pattern with the horizon (Figure 12, left panel). In particular, they start being significant around 5 to 6 years, pointing to the existence of components of the market return and illiquidity process - whose periodicity is lower than the business cycle - which are likely linked by a predictive relation. As shown by Bandi et al. (2012), these details can be extracted directly. Thus, predictability may be verified on the details themselves rather on low-pass (aggregation) filters as in our current approach. Being the paper's focus on EXIT, on its theoretical justification, and on its empirical potential, rather than on illiquidity pricing per se, we leave the latter approach for future work.

Aggregation does not lead to spurious increasing patterns. Bandi et al. (2012) show that, in the absence of low-frequency predictive relations at the level of individual details of the regressand and regressor, no increasing pattern would be found. Similarly, if aggregation had a mechanical impact leading to increasing patterns, contemporaneous aggregation would lead to similar outcomes, and it fails to do so. In Figure 12, right panel, we regress long-run returns $p_{t+h} - p_t$ on EXIT aggregates over the same horizon, $\text{EXIT}_{t+1:t+h}$. The increasing pattern is now hardly statistically significant. Over the long run, forward/backward aggregation

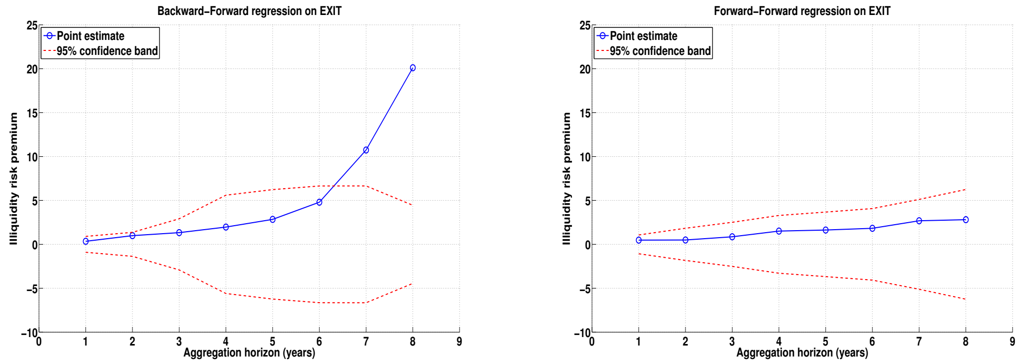

Figure 12: Left panel: we regress forward aggregated returns on backward aggregated values of EXIT, over horizons longer than one year. Right panel: we regress forward aggregated returns on forward aggregated values of EXIT.

yields slope estimates whose magnitude is two to four times that obtained when aggregating contemporaneously.

7 Further discussion

7.1 EXITs (standardized EXIT)

EXIT is designed to capture an important aspect of the price formation process under illiquidity (when there is a role for asymmetries in information), namely sluggishness in the price adjustments. In this sense, its reliance on transaction prices alone makes it an easy measure to implement.

There is, however, a second aspect of the assumed frictional alternative in Section 2 which EXIT - as defined above - does not capture: lack of trading. As illustrated, when transaction costs are high, the informed agents may not find it convenient to trade. In the no-trade area, prices do not move, thereby leading to repetitions of the same price when sampling is conducted by imputing the previous tick in the absence of an actual price update. Stale recorded prices are, therefore, associated with a small number of transactions.

To account for *both* of the implications of the price formation model in Section 2 (i.e. price staleness as well as lack of trades), EXIT may be re-defined as the proposed measure standardized by the number of transactions (EXITs):

$$\text{EXITs} = \frac{\frac{1}{T}\sum_{i=1}^{n}\left(\Delta_n 1_{\{|\widetilde{p}_{t_i}-\widetilde{p}_{t_{i-1}}|\le\xi_n\}} - \sqrt{\frac{2}{\pi}}\frac{\xi_n\Delta_n}{\sigma_\varepsilon}\right)}{\#\text{ of transactions over } [0,T]}.$$

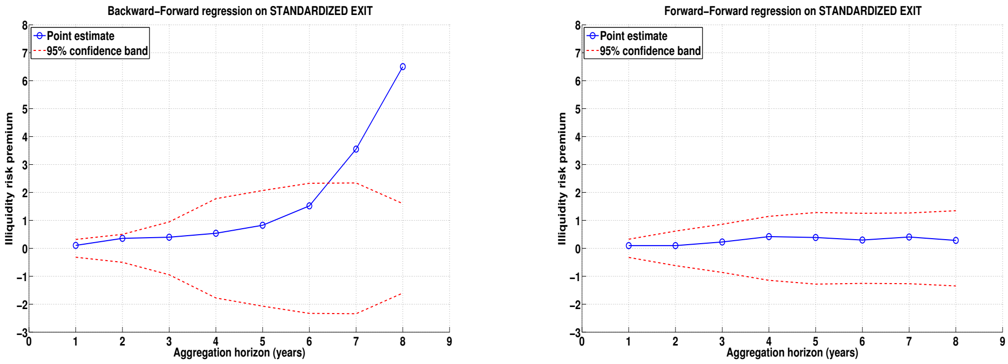

Figure 13: Left panel: we regress forward aggregated returns on backward aggregated values of EXITs (standardized EXIT), over horizons longer than one year. Right panel: we regress forward aggregated returns on forward aggregated values of EXITs.

EXITs is higher, the slower the price adjustments and the smaller the number of transactions for each period, thereby capturing two important aspects of the assumed alternative. Empirically, the measure captures the fact that slow price adjustments in the presence of a large number of trades may be less informative about lack of liquidity than slow price adjustments in the presence of a small number of trades. This is, of course, done at the cost of additional information being brought into the formulation of the original measure (the number of trades). While this information is often readily available, it does add an extra layer to a measure (like EXIT) whose reliance on transaction prices only is appealing.

The bias adjustment in the numerator $\left(\sqrt{\frac{2}{\pi}}\frac{\xi_n\Delta_n}{\sigma_\varepsilon}\right)$ is consistent with that in Eq. (2.5). There, the bias adjustment was derived under a null hypothesis in which observed prices are Ito semimartingales. Here, the adjustment is derived under a null hypothesis in which there is a role for recorded price deviations around the underlying equilibrium Ito semimartingale. The latter approach is, as discussed throughout, consistent with the alternative in Section 2 when setting the probability of arrival of informed traders $\mathcal{I}$ equal to zero.

For the S&P 500 futures data used in the previous section, we find that the dynamic properties of EXIT and EXITs are very similar. For illustration, Figure 13 evaluates long-run liquidity premia using EXITs yielding outcomes that are analogous to those presented in the previous section.

7.2 The role of asymmetric information

This paper provides a formal econometric test for the presence of asymmetric information which was applied earlier to market returns. Being asymmetric information a fundamental feature of modern market microstructure theory, a statistical take on this issue appears warranted.

The presence of asymmetric information, leading to information-based trading or lack thereof, justifies using EXIT as an illiquidity proxy. Lack of trading, due to high execution costs, on the part of noise traders (something which is outside of the model in Section 2) would also yield similar effects (and add to lack of information-based trading), thereby providing an even stronger empirical motivation for using EXIT as a measure of illiquidity.

Time-variation in asymmetric information, in addition to the *presence* of asymmetric information, should however lead to a broader interpretation of the measure. Should asymmetric information be time-varying, rather than stable or slow-varying as assumed throughout, EXIT could be more broadly interpreted as a *friction* (illiquidity and asymmetric information) proxy, rather than solely as an illiquidity proxy correlated with execution costs. As shown in Section 2, EXIT would in fact increase both with increases in c and with increases in $\mathcal{I}$.

The link between illiquidity and asymmetric information - fundamentally connected phenomena with many facets and hard-to-pin-down features - is, importantly, not specific to our approach. Rather, being information a basic aspect of modern market microstructure theory, it is something that impacts a large number of illiquidity measures. Consider the high-frequency benchmark in Eq. (3.2) of this paper or, equivalently, in Eq. (2) of Goyenko et al. (2009), as an example. In this case, the *presence* of information asymmetries forces to the choice of a future trade (p_{k+j}) to measure execution costs correcly. *Time-variation* in these information asymmetries should lead to a choice of j (the time lag) that is also time-varying, something which is generally unaccounted for.

In sum, contrary to information-based trading (which ought to be time-varying and correlated with the size of execution costs), the extent of asymmetries in information or, somewhat equivalently, the size of the pool of informed traders is typically assumed to be constant, or slow-moving, over time. Whether this maintained assumption in the liquidity literature is a valid approximation is an issue for future work. The framework that we propose is one way to make the connection between illiquidity and asymmetric information explicit and could shed light on aspects of the former (or of its measurements) should the latter be found to also be time-varying.

8 Conclusions

We introduce a novel stochastic quantity, named *excess idle time* or EXIT, measuring sluggishness in the updates to transaction prices. Staleness in the price adjustments leads to large values of our proposed measure, small values being coherent with erratic price behaviour. We show that price sluggishness is readily delivered by models in which a role is given to asymmetries in information, as well as to the magnitude of execution costs, in determining asset prices. This is a natural alternative hypothesis in our framework, yielding stickier transaction prices - due to lack of informed-based trading - when execution costs are higher (for any positive level of asymmetric information). A coherent (with this alternative) null hypothesis is one in which erratic price behaviour is induced by noise trading and the learning of market makers about an unpredictable semimartingale efficient price process. This natural null hypothesis assumes that the observed price process is driven by a *noise-contaminated* semimartingale, an assumption under which a complete asymptotic theory is derived and EXIT is justified formally.

EXIT is easily-computable based on high-frequency transaction prices only. Economically, its magnitude provides information about the extent of frictions (illiquidity and asymmetric information) in the determination of observed prices. We show that, for any given level of asymmetric information, EXIT has the potential to be more correlated with (nominal and effective) execution costs than well-known benchmarks, particularly in the presence of clustering in order flow, an important empirical regularity.

Consistent with this observation, using EXIT has an illiquidity proxy, we provide evidence for short-term and long-term compensations for illiquidity risk in excess market returns. This latter study should solely be viewed as being illustrative of the potential of the new measure. Such a study will be broadened in scope to evaluate, among other issues, compensation for cross-sectional (systematic and idiosyncratic) illiquidity risk.

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A Appendix: Proofs

Lemma 1. *Write*

$$Y_\Delta - Y_0 = X_\Delta - X_0 + \underbrace{\eta_\Delta - \eta_0}_{\varepsilon_\Delta},$$

where the ηs are iid mean zero normal shocks (independent of the Xs) with variance σ_η^2. Then, when X satisfies Assumption 1, the pdf of $Y_\Delta = \frac{X_\Delta+\varepsilon}{\sigma_0\sqrt{\Delta}}$, conditional on time 0 information, can be expressed as

$$\phi_{\frac{\mu_0\Delta-\eta_0}{\sigma_0\sqrt{\Delta}},1+\frac{\sigma_\eta^2}{\sigma_0^2\Delta}}(x)\left(1+\frac{c\sqrt{\Delta}}{\left(1+\frac{\sigma_\eta^2}{\sigma_0^2\Delta}\right)^{3/2}}\left(\left[\frac{x-\left(\frac{\mu_0\Delta-\eta_0}{\sigma_0\sqrt{\Delta}}\right)}{\sqrt{1+\frac{\sigma_\eta^2}{\sigma_0^2\Delta}}}\right]^3-3\left[\frac{x-\left(\frac{\mu_0\Delta-\eta_0}{\sigma_0\sqrt{\Delta}}\right)}{\sqrt{1+\frac{\sigma_\eta^2}{\sigma_0^2\Delta}}}\right]\right)+o(\sqrt{\Delta})\right),$$

where $\phi_{\mu,\sigma^2}(x)$ is the normal density with mean μ and variance σ^2 and c is some constant.

Proof of Lemma 1. Assume, without loss of generality, $Y_0 = 0 = X_0$. Define the standardized quantity

$$Z_\Delta = \frac{X_\Delta - \mu_0\Delta}{\sigma_0\sqrt{\Delta}}.$$

Notice that the characteristic function of the standardized process

$$\widetilde{Y}_\Delta = \frac{X_\Delta + \varepsilon_\Delta}{\sigma_0\sqrt{\Delta}}$$

can be factorized as follows

$$\Psi_{\widetilde{Y}_\Delta}(t) \quad = \quad \mathrm{E}_0\left[e^{it\frac{X_\Delta+\varepsilon_\Delta}{\sigma_0\sqrt{\Delta}}}\right] = \mathrm{E}_0\left[e^{it\frac{X_\Delta-\mu_0\Delta}{\sigma_0\sqrt{\Delta}}}e^{it\frac{\mu_0\Delta}{\sigma_0\sqrt{\Delta}}}e^{it\frac{\varepsilon_\Delta}{\sigma_0\sqrt{\Delta}}}\right]$$

$$
\begin{aligned}
&= e^{it\frac{\mu_0\Delta-\eta_0}{\sigma_0\sqrt{\Delta}}}\mathrm{E}_0\left[e^{it\frac{X_\Delta-\mu_0\Delta}{\sigma_0\sqrt{\Delta}}}\right]\mathrm{E}_0\left[e^{it\frac{\eta_\Delta}{\sigma_0\sqrt{\Delta}}}\right] \\
&= e^{it\frac{\mu_0\Delta-\eta_0}{\sigma_0\sqrt{\Delta}}}\mathrm{E}_0\left[e^{it\frac{X_\Delta-\mu_0\Delta}{\sigma_0\sqrt{\Delta}}}\right]e^{-\frac{t^2}{2}\left(\frac{\sigma_\eta^2}{\sigma_0^2\Delta}\right)}.
\end{aligned}
$$

Using Bandi and Renò (2013), write

$$
\mathrm{E}_0\left[e^{it\frac{X_\Delta-\mu_0\Delta}{\sigma_0\sqrt{\Delta}}}\right] = e^{-\frac{t^2}{2}}(1+c\sqrt{\Delta}(it)^3+o(\sqrt{\Delta})),
$$

where c is a suitable constant. Finally,

$$
\Psi_{\widetilde{Y}_\Delta}(t) = e^{it\frac{\mu_0\Delta-\eta_0}{\sigma_0\sqrt{\Delta}}-\frac{t^2}{2}\left(1+\frac{\sigma_\eta^2}{\sigma_0^2\Delta}\right)}(1+c\sqrt{\Delta}(it)^3+o(\sqrt{\Delta})).
$$

Now, write the conditional density of $\widetilde{Y}_\Delta$ as

$$
\begin{aligned}
f_{\widetilde{Y}_\Delta}(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-itx}\Psi_{\widetilde{Y}_\Delta}(t)dt \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-itx}\left[e^{it\frac{\mu_0\Delta-\eta_0}{\sigma_0\sqrt{\Delta}}-\frac{t^2}{2}\left(1+\frac{\sigma_\eta^2}{\sigma_0^2\Delta}\right)}\left(1+c\sqrt{\Delta}(it)^3+o(\sqrt{\Delta})\right)\right]dt \\
&= \phi_{\frac{\mu_0\Delta-\eta_0}{\sigma_0\sqrt{\Delta}},1+\frac{\sigma_\eta^2}{\sigma_0^2\Delta}}(x)\left(1+o(\sqrt{\Delta})\right)+\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-itx}e^{it\frac{\mu_0\Delta-\eta_0}{\sigma_0\sqrt{\Delta}}-\frac{t^2}{2}\left(1+\frac{\sigma_\eta^2}{\sigma_0^2\Delta}\right)}c\sqrt{\Delta}(it)^3dt \\
&= \phi_{\frac{\mu_0\Delta-\eta_0}{\sigma_0\sqrt{\Delta}},1+\frac{\sigma_\eta^2}{\sigma_0^2\Delta}}(x)\left(1+o(\sqrt{\Delta})\right)-\frac{c\sqrt{\Delta}}{2\pi}\frac{\partial^3}{\partial^3x}\int_{-\infty}^{\infty}e^{-itx}e^{it\frac{\mu_0\Delta-\eta_0}{\sigma_0\sqrt{\Delta}}-\frac{t^2}{2}\left(1+\frac{\sigma_\eta^2}{\sigma_0^2\Delta}\right)}dt \\
&= \phi_{\frac{\mu_0\Delta-\eta_0}{\sigma_0\sqrt{\Delta}},1+\frac{\sigma_\eta^2}{\sigma_0^2\Delta}}(x)\left(1+o(\sqrt{\Delta})\right)-c\sqrt{\Delta}\frac{\partial^3}{\partial^3x}\phi_{\frac{\mu_0\Delta-\eta_0}{\sigma_0\sqrt{\Delta}},1+\frac{\sigma_\eta^2}{\sigma_0^2\Delta}}(x) \\
&= \phi_{\frac{\mu_0\Delta-\eta_0}{\sigma_0\sqrt{\Delta}},1+\frac{\sigma_\eta^2}{\sigma_0^2\Delta}}(x)\left(1+\frac{c\sqrt{\Delta}}{\left(1+\frac{\sigma_\eta^2}{\sigma_0^2\Delta}\right)^{3/2}}\left(\left[\frac{x-\left(\frac{\mu_0\Delta-\eta_0}{\sigma_0\sqrt{\Delta}}\right)}{\sqrt{1+\frac{\sigma_\eta^2}{\sigma_0^2\Delta}}}\right]^3-3\left[\frac{x-\left(\frac{\mu_0\Delta-\eta_0}{\sigma_0\sqrt{\Delta}}\right)}{\sqrt{1+\frac{\sigma_\eta^2}{\sigma_0^2\Delta}}}\right]\right)+o(\sqrt{\Delta})\right),
\end{aligned}
$$

which proves the stated result.

Lemma 2. *For $\beta > 0$, and for a càdlàg process $g_t > 0$, we have*

$$
\left(\frac{n}{T}\right)^{\beta-1}\sum_{i=1}^{n}\frac{1}{g_{t_{i-1}}}\Delta_{n,i}^{\beta}\xrightarrow{p}\int_0^T\frac{1}{g_s}H_\beta^{'}(s)ds
$$

as $n \to \infty$.

Proof of Lemma 2. The proof follows the same line as Lemma A.1 of Mancini et al. (2012).

Lemma 3. Let $\beta > 0$. For an IID process X and a measurable function $f(.)$ such that $f(X)$ has finite variance, we have, as $n \to \infty$,

$$
\left(\frac{n}{T}\right)^{\beta-1}\sum_{i=1}^{n}f(X_i)\Delta_{n,i}^{\beta}\xrightarrow{p}\mathrm{E}(f(X))H_\beta(T).
$$

Proof. Immediate since

$$\underbrace{\left(\frac{n}{T}\right)^{\beta-1}\sum_{i=1}^{n}\left[f(X_i)-\mathrm{E}(f(X))\right]\Delta_{n,i}^{\beta}}_{\leq O_p\left(\sqrt{\left(\frac{n}{T}\right)^{2\beta-2}\sum_{i=1}^{n}\Delta_{n,i}^{2\beta}}\right)=O_p\left(\sqrt{\frac{1}{n}}\right)}+\underbrace{\mathrm{E}(f(X))\left(\frac{n}{T}\right)^{\beta-1}\sum_{i=1}^{n}\Delta_{n,i}^{\beta}}_{\to\mathrm{E}(f(X))H_\beta(T)\text{ by Lemma 2}}$$

using Chebyshev's inequality and the bound on the variance of $f(X)$.

Proof of Theorem 1 and Theorem 2. Denote by $X_t = p_t^e$ and $Y_t = \widetilde{p}_t$, so that $Y_\Delta - Y_0 = X_\Delta - X_0 + \varepsilon_\Delta$ where $\varepsilon_\Delta = \eta_\Delta - \eta_0$. The proof is divided into three parts.

Part 1: preliminary estimates.

Write $\sigma(\Delta) = \left(1+\frac{\sigma_\eta^2}{\sigma_0^2\Delta}\right)^{1/2}$. Notice that

$$\begin{aligned}\sigma^2(\Delta) &= \frac{\sigma_0^2\Delta+\sigma_\eta^2}{\sigma_0^2\Delta}\to\infty \text{ as } \Delta\to 0 \text{ if } \sigma_\eta^2\neq 0,\\ \sigma^2(\Delta) &= 1 \text{ if } \sigma_\eta^2 = 0.\end{aligned}$$

Ignoring the smaller order term, Lemma 1 implies that the following conditional probability can be written as:

$$\begin{aligned}\psi_0 &= E_0\left[1_{\{|Y_\Delta-Y_0|\leq\xi_n\}}\right] = P_0\left(\left|\frac{Y_\Delta-Y_0}{\sigma_0\sqrt{\Delta}}\right|\leq\frac{\xi_n}{\sigma_0\sqrt{\Delta}}\right)\\ &= \int_{-\xi_n/\sigma_0\sqrt{\Delta}}^{\xi_n/\sigma_0\sqrt{\Delta}}\phi_{\frac{\mu_0\Delta-\eta_0}{\sigma_0\sqrt{\Delta}},\sigma^2(\Delta)}(x)\left(1+\frac{c\sqrt{\Delta}}{\sigma^3(\Delta)}\left(\left[\frac{x-\left(\frac{\mu_0\Delta-\eta_0}{\sigma_0\sqrt{\Delta}}\right)}{\sigma(\Delta)}\right]^3-3\left[\frac{x-\left(\frac{\mu_0\Delta-\eta_0}{\sigma_0\sqrt{\Delta}}\right)}{\sigma(\Delta)}\right]\right)\right)dx\\ &= \int_{\frac{-\xi_n-\mu_0\Delta+\eta_0}{(\sigma_0\sqrt{\Delta})\sigma(\Delta)}}^{\frac{\xi_n-\mu_0\Delta+\eta_0}{(\sigma_0\sqrt{\Delta})\sigma(\Delta)}}\phi_{0,1}(y)\left(1+\frac{c\sqrt{\Delta}}{\sigma^3(\Delta)}\left(y^3-3y\right)\right)dy.\end{aligned}$$

This implies that, ignoring the smaller order terms again,

$$\begin{aligned}\psi_0 &= \frac{1}{\sqrt{2\pi}}\int_{\frac{-\xi_n-\mu_0\Delta+\eta_0}{(\sigma_0\sqrt{\Delta})\sigma(\Delta)}}^{\frac{\xi_n-\mu_0\Delta+\eta_0}{(\sigma_0\sqrt{\Delta})\sigma(\Delta)}} e^{-\frac{1}{2}y^2}\left(1+\frac{c\sqrt{\Delta}}{\sigma^3(\Delta)}\left(y^3-3y\right)\right)dy\\ &= \frac{1}{\sqrt{2\pi}}\int_{\frac{-\xi_n-\mu_0\Delta+\eta_0}{(\sigma_0\sqrt{\Delta})\sigma(\Delta)}}^{\frac{\xi_n-\mu_0\Delta+\eta_0}{(\sigma_0\sqrt{\Delta})\sigma(\Delta)}}(1-\frac{1}{2}y^2+\frac{1}{8}y^4+...)\left(1+\frac{c\sqrt{\Delta}}{\sigma^3(\Delta)}\left(y^3-3y\right)\right)dy\\ &= \frac{1}{\sqrt{2\pi}}\int_{\frac{-\xi_n-\mu_0\Delta+\eta_0}{(\sigma_0\sqrt{\Delta})\sigma(\Delta)}}^{\frac{\xi_n-\mu_0\Delta+\eta_0}{(\sigma_0\sqrt{\Delta})\sigma(\Delta)}}\left(1-\frac{1}{2}y^2+\frac{1}{8}y^4+...\right)dy\\ &= \frac{1}{\sqrt{2\pi}}\left\{\frac{2\xi_n}{\sigma(\Delta)\sqrt{\sigma_0^2\Delta}}-\left[\frac{1}{1!2^1}\frac{1}{3}y^3\right]_{\frac{-\xi_n-\mu_0\Delta+\eta_0}{(\sigma_0\sqrt{\Delta})\sigma(\Delta)}}^{\frac{\xi_n-\mu_0\Delta+\eta_0}{(\sigma_0\sqrt{\Delta})\sigma(\Delta)}}+\left[\frac{1}{2!2^2}\frac{1}{5}y^5\right]_{\frac{-\xi_n-\mu_0\Delta+\eta_0}{(\sigma_0\sqrt{\Delta})\sigma(\Delta)}}^{\frac{\xi_n-\mu_0\Delta+\eta_0}{(\sigma_0\sqrt{\Delta})\sigma(\Delta)}}+..\right\}\end{aligned}$$

$$= \frac{\xi_n}{\sqrt{2\,\pi}} \left\{ \frac{2}{\sigma(\Delta)\sqrt{\sigma_0^2 \Delta}} + \sum_{k=1}^{\infty} (-)^k \frac{\left[(\xi_n - \mu_0 \Delta + \eta_0)^{2\,k+1} - (-\xi_n - \mu_0 \Delta + \eta_0)^{2\,k+1} \right]}{k!\, 2^k \,(2\,k+1) \xi_n \left[(\sigma_0 \sqrt{\Delta}) \sigma(\Delta) \right]^{2\,k+1}} \right\} \tag{A.1}$$

Now notice that, for any integer $k \geq 1$,

$$\begin{aligned}
(\xi_n - \mu_0 \Delta + \eta_0)^{2\,k+1} - (-\xi_n - \mu_0 \Delta + \eta_0)^{2\,k+1} &= 2 \sum_{j=1,3,\ldots,2k+1} \binom{2k+1}{j} \xi_n^j (\eta_0 - \mu_0 \Delta)^{2k+1-j} \\
&= (2\,k+1)\; \eta_0^{2\,k}\, \xi_n + O_p\left(\xi_n^3 + \xi_n \Delta \right) + R_{n,k}
\end{aligned} \tag{A.2}$$

so that we can write:

$$\begin{aligned}
\psi_0 &= \frac{\xi_n}{\sqrt{2\,\pi}} \left\{ \frac{2}{\sigma(\Delta)\sqrt{\sigma_0^2 \Delta}} + \sum_{k=1}^{\infty} \frac{(-)^k}{(k!\, 2^k)} \frac{1}{\left[(\sigma_0 \sqrt{\Delta}) \sigma(\Delta) \right]^{2\,k+1}} \frac{2\,\eta_0^{2\,k}\, \xi_n + O_p\left(\xi_n^3 + \xi_n \Delta \right) + R_{n,k}}{\xi_n} \right\} \\
&= \frac{\xi_n}{\sqrt{2\,\pi}} \left\{ \frac{2}{\sigma(\Delta)\sqrt{\sigma_0^2 \Delta}} + \sum_{k=1}^{\infty} \frac{(-)^k}{(k!\, 2^k)} \frac{1}{\left[(\sigma_0 \sqrt{\Delta}) \sigma(\Delta) \right]^{2\,k+1}} \left[2\,\eta_0^{2\,k} + \frac{O_p\left(\xi_n^3 + \xi_n \Delta \right) + R_{n,k}}{\xi_n} \right] \right\} \\
&= \frac{2\,\xi_n}{\sqrt{2\,\pi}\, \sigma(\Delta)\sqrt{\sigma_0^2 \Delta}} \left\{ 1 + \sum_{k=1}^{\infty} \frac{(-)^k}{(k!\, 2^k)} \frac{1}{\left[(\sigma_0 \sqrt{\Delta}) \sigma(\Delta) \right]^{2\,k}} \left[\eta_0^{2\,k} + \frac{O_p\left(\xi_n^3 + \xi_n \Delta \right) + R_{n,k}}{\xi_n} \right] \right\} \\
&= \frac{2\,\xi_n}{\sigma(\Delta)\, \sqrt{2\,\pi\, \sigma_0^2 \Delta}} \left\{ 1 + \sum_{k=1}^{\infty} \frac{(-)^k}{(k!\, 2^k)} \frac{1}{\left[(\sigma_0 \sqrt{\Delta}) \sigma(\Delta) \right]^{2\,k}} \eta_0^{2\,k} \right. \\
&\quad \left. + \frac{O_p\left(\xi_n^3 + \xi_n \Delta \right)}{\xi_n} \sum_{k=1}^{\infty} \frac{(-)^k}{(k!\, 2^k)} \frac{1}{\left[(\sigma_0 \sqrt{\Delta}) \sigma(\Delta) \right]^{2\,k}} + \frac{1}{\xi_n} \sum_{k=1}^{\infty} \frac{(-)^k}{(k!\, 2^k)} \frac{R_{n,k}}{\left[(\sigma_0 \sqrt{\Delta}) \sigma(\Delta) \right]^{2\,k}} \right\} \\
&= \frac{2\,\xi_n}{\sigma(\Delta)\, \sqrt{2\,\pi\, \sigma_0^2 \Delta}} \left\{ e^{-\frac{1}{2} \left(\frac{\eta_0}{(\sigma_0 \sqrt{\Delta}) \sigma(\Delta)} \right)^2} + \frac{O_p\left(\xi_n^3 + \xi_n \Delta \right)}{\xi_n} \sum_{k=1}^{\infty} \frac{(-)^k}{(2\,k)! 2^k} \frac{1}{\left[(\sigma_0 \sqrt{\Delta}) \sigma(\Delta) \right]^{2\,k}} \right. \\
&\quad \left. + \frac{1}{\xi_n} \sum_{k=1}^{\infty} \frac{(-)^k}{(k!\, 2^k)} \frac{R_{n,k}}{\left[(\sigma_0 \sqrt{\Delta}) \sigma(\Delta) \right]^{2\,k}} \right\}.
\end{aligned}$$

Now, the last term is negligible since $R_{n,k}$ is at most proportional to $(2k+1)!/(k-1)!(k+1)!$. Thus, the summation containing it converges and is asymptotically negligible with respect to the other terms. Finally, if $\sigma_\eta \neq 0$, we have $(\sigma_0 \sqrt{\Delta}) \sigma(\Delta) \sim \sigma_\eta$ and

$$\psi_0 = \frac{2\,\xi_n}{\sqrt{2\,\pi\, \sigma_\eta^2}}\, e^{-\frac{1}{2} \left(\frac{\eta_0}{\sigma_\eta} \right)^2} + O_p\left(\xi_n^3 + \xi_n \Delta \right). \tag{A.3}$$

If, instead, $\sigma_\eta = 0$ we have $\sigma(\Delta) = 1$ which implies, from Eq. (A.1),

$$\psi_0 = \sqrt{\frac{2}{\pi}} \frac{\xi_n\, \Delta^{-1/2}}{\sigma_0} + O_p\left(\left(\xi_n\, \Delta^{-1/2} \right)^3 \right). \tag{A.4}$$

Part 2: the proof under the null.

In what follows, for conciseness, we write $\Delta_{n,i}$ as Δ_i. Set $p^F = 0$. We start with the sum of the first conditional moments. Write, in the case $\eta_i = 0$ and using Eq. (A.4),

$$\begin{aligned}
\frac{1}{T}\sum_{i=1}^{n} E_{i-1}\left[(t_i - t_{i-1})1_{\left\{\left|\frac{Y_i - Y_{i-1}}{\sigma_{i-1}\sqrt{\Delta_i}}\right| \le \frac{\xi_n}{\sigma_{i-1}\sqrt{\Delta_i}}\right\}}\right] &= \frac{1}{T}\sum_{i=1}^{n}(t_i - t_{i-1})P_{i-1}\left\{\left|\frac{Y_i - Y_{i-1}}{\sigma_{i-1}\sqrt{\Delta_i}}\right| \le \frac{\xi_n}{\sigma_{i-1}\sqrt{\Delta_i}}\right\} \\
&= \frac{1}{T}\sum_{i=1}^{n}(t_i - t_{i-1})\left[\sqrt{\frac{2}{\pi}}\frac{\xi_n\,\Delta_i^{-1/2}}{\sigma_0} + O_p\left(\left(\xi_n\,\Delta_i^{-1/2}\right)^3\right)\right] \\
&\overset{p}{\sim} \left(\xi_n\sqrt{n}\right)\sqrt{\frac{2}{\pi}}T^{-\frac{3}{2}}\int_0^T \frac{1}{\sigma_s}H'_{1/2}(s)ds,
\end{aligned}$$

where Lemma 2 was used. In the case $\eta_i \neq 0$, we use Eq. (A.3), Lemma 3 and write

$$\begin{aligned}
\frac{1}{T}\sum_{i=1}^{n} E_{i-1}\left[(t_i - t_{i-1})1_{\left\{\left|\frac{Y_i - Y_{i-1}}{\sigma_{i-1}\sqrt{\Delta_i}}\right| \le \frac{\xi_n}{\sigma_{i-1}\sqrt{\Delta_i}}\right\}}\right] &= \frac{1}{T}\sum_{i=1}^{n}(t_i - t_{i-1})\left(\frac{2e^{-\frac{1}{2}\left(\frac{\eta_{i-1}}{\sigma_\eta}\right)^2}\xi_n}{\sqrt{2\pi\sigma_\eta^2}} + O_p(\xi_n^3 + \xi_n\Delta_i)\right) \\
&\overset{p}{\sim} \frac{1}{T}\sum_{i=1}^{n}(t_i - t_{i-1})\frac{2E\left[e^{-\frac{1}{2}\left(\frac{\eta_{i-1}}{\sigma_\eta}\right)^2}\right]\xi_n}{\sqrt{2\pi\sigma_\eta^2}} \\
&\overset{p}{\sim} 2\xi_n\left(T^{-1}\int_0^T H'_1(s)ds\right)\frac{1}{\sqrt{2\pi\sigma_\eta^2}}\frac{1}{\sqrt{2}} \\
&= \sqrt{\frac{2}{\pi}}\frac{\xi_n}{\sigma_\varepsilon},
\end{aligned}$$

where $\sigma_\varepsilon^2 = 2\sigma_\eta^2$.

Write the sum of the second conditional moments as

$$\begin{aligned}
\frac{1}{T^2}\sum_{i=1}^{n} E_{i-1}\left[(t_i - t_{i-1})^2 1^2_{\left\{\left|\frac{Y_i - Y_{i-1}}{\sigma_{i-1}\sqrt{\Delta_i}}\right| \le \frac{\xi_n}{\sigma_{i-1}\sqrt{\Delta_i}}\right\}}\right] &= \frac{1}{T^2}\sum_{i=1}^{n}(t_i - t_{i-1})^2 P_{i-1}\left\{\left|\frac{Y_i - Y_{i-1}}{\sigma_{i-1}\sqrt{\Delta_i}}\right| \le \frac{\xi_n}{\sigma_{i-1}\sqrt{\Delta_i}}\right\} \\
&= \frac{1}{T^2}\sum_{i=1}^{n}(t_i - t_{i-1})^2\frac{2\xi_n}{\sigma(\Delta_i)\sqrt{2\pi\sigma_{i-1}^2\Delta_i}} + o_p(1) \\
&\overset{p}{\sim} \left(\xi_n/\sqrt{n}\right)\sqrt{\frac{2}{\pi}}T^{-\frac{3}{2}}\int_0^T \frac{1}{\sigma_s}H'_{3/2}(s)ds,
\end{aligned}$$

if $\eta_i = 0$. Otherwise, using again Lemma 3,

$$\begin{aligned}
\frac{1}{T^2}\sum_{i=1}^{n} E_{i-1}\left[(t_i - t_{i-1})^2 1_{\left\{\left|\frac{Y_i - Y_{i-1}}{\sigma_{i-1}\sqrt{\Delta_i}}\right| \le \frac{\xi_n}{\sigma_{i-1}\sqrt{\Delta_i}}\right\}}\right] &\overset{p}{\sim} \frac{1}{T^2}\sum_{i=1}^{n}(t_i - t_{i-1})^2\frac{2e^{-\frac{1}{2}\left(\frac{\eta_{i-1}}{(\sigma_{i-1}\sqrt{\Delta_i})\sigma(\Delta_i)}\right)^2}\xi_n}{\sigma(\Delta_i)\sqrt{2\pi\sigma_{i-1}^2\Delta_i}} \\
&\overset{p}{\sim} \frac{1}{T^2}\sum_{i=1}^{n}(t_i - t_{i-1})^2\,\frac{2e^{-\frac{1}{2}\left(\frac{\eta_{i-1}}{\sigma_\eta}\right)^2}\xi_n}{\sqrt{2\pi\sigma_\eta^2}} \\
&\overset{p}{\sim} \frac{1}{T^2}\sum_{i=1}^{n}(t_i - t_{i-1})^2\,\frac{2E\left[e^{-\frac{1}{2}\left(\frac{\eta_{i-1}}{\sigma_\eta}\right)^2}\right]\xi_n}{\sqrt{2\pi\sigma_\eta^2}} \\
\left\{\sigma_\epsilon = \sqrt{2}\,\sigma_\eta \Rightarrow \sqrt{2}\,\sigma_\epsilon = 2\,\sigma_\eta\right\} &= 2\,\frac{\xi_n}{n}\,T^{-1}\left(\int_0^T H'_2(s)ds\right)\frac{1}{\sqrt{2\pi}\sigma_\varepsilon}
\end{aligned}$$

$$= \quad (\xi_n/n)\; T^{-1}\sqrt{\frac{2}{\pi}}\frac{H_2(T)}{\sigma_\varepsilon}.$$

We now turn to the limiting distribution. Write

$$\widetilde{\phi}_{n,i} = \frac{1}{T}\left((t_i - t_{i-1})1_{\left\{\left|\frac{Y_i-Y_{i-1}}{\sigma_{i-1}\sqrt{\Delta_i}}\right|\le\frac{\xi_n}{\sigma_{i-1}\sqrt{\Delta_i}}\right\}} - \sqrt{\frac{2}{\pi}}\frac{\xi_n\sqrt{\Delta_i}}{\sigma_{i-1}}\right).$$

and

$$\widetilde{\phi}'_{n,i} = \frac{1}{T}\left((t_i - t_{i-1})1_{\left\{\left|\frac{Y_i-Y_{i-1}}{\sigma_{i-1}\sqrt{\Delta_i}}\right|\le\frac{\xi_n}{\sigma_{i-1}\sqrt{\Delta_i}}\right\}} - \sqrt{\frac{2}{\pi}}\frac{\xi_n\Delta_i}{\sigma_\varepsilon}\right).$$

If $\eta_i = 0$, then

$$\begin{aligned}
\left(\frac{\sqrt{n}}{\xi_n}\right)^{1/2}\sum_{i=1}^n E_{i-1}\left[\widetilde{\phi}_{n,i}\right] &= \left(\frac{\sqrt{n}}{\xi_n}\right)^{1/2} T^{-1}\sum_{i=1}^n\left[\Delta_i\psi_{i-1} - \sqrt{\frac{2}{\pi}}\frac{\xi_n\sqrt{\Delta_i}}{\sigma_{i-1}}\right]\\
&= \left(\frac{\sqrt{n}}{\xi_n}\right)^{1/2} T^{-1}\sum_{i=1}^n\left[\sqrt{\frac{2}{\pi}}\frac{\xi_n}{\sigma_{i-1}}\Delta_i^{1/2} + O_p\left(\xi_n^3\,\Delta_i^{-1/2}\right) - \sqrt{\frac{2}{\pi}}\frac{\xi_n\,\Delta_i^{1/2}}{\sigma_{i-1}}\right]\\
&= n^{1/4}\,\xi_n^{-1/2}\,n\,O_p\left(\xi_n^3\,n^{1/2}\right)\\
&= O_p\left(n^{7/4}\,\xi_n^{5/2}\right) = O_p\left(n^{7/10}\xi_n\right)^{5/2} \to 0,
\end{aligned}$$

if $n^{7/10}\xi_n \to 0$. If $\eta_i \neq 0$, write

$$\Psi_n = \left(\frac{n}{\xi_n}\right)^{1/2}\sum_{i=1}^n E_{i-1}\left[\widetilde{\phi}'_{n,i}\right].$$

We have

$$\Psi_n^2 = \left(\frac{n}{\xi_n}\right)\left[\sum_{i=1}^n\left(E_{i-1}\left[\widetilde{\phi}'_{n,i}\right]\right)^2 + 2\sum_{i>j}E_{i-1}\left[\widetilde{\phi}'_{n,i}\right]E_{j-1}\left[\widetilde{\phi}'_{n,j}\right]\right].$$

Now, using (A.3), we obtain

$$E_{i-1}\left[\widetilde{\phi}'_{n,i}\right] = \sqrt{\frac{2}{\pi}}\frac{\xi_n\Delta_i}{\sigma_\eta}\left(e^{-\frac{1}{2}\left(\frac{\eta_{i-1}}{\sigma_\eta}\right)^2} - \frac{1}{\sqrt{2}}\right) + O_p(\Delta_i\xi_n^3)$$

so that, using the independence of the η_is, we have

$$E[\Psi_n^2] = \left(\frac{n}{\xi_n}\right)O\left(n\left(\frac{\xi_n}{n}\right)^2 + n^2\left(\frac{\xi_n^3}{n}\right)^2\right).$$

This proves that $\Psi_n = O_p\left(\xi_n^{5/2}n^{1/2}\right)$.

If $\eta_i = 0$, write

$$\begin{aligned}
\left(\frac{\sqrt{n}}{\xi_n}\right)\sum_{i=1}^n E_{i-1}\left[\widetilde{\phi}_{n,i}^2\right] &= \left(\frac{\sqrt{n}}{\xi_n}\right)\frac{1}{T^2}\sum_{i=1}^n(t_i - t_{i-1})^2 P_{i-1}\left(\left|\frac{Y_i - Y_{i-1}}{\sigma_{i-1}\sqrt{\Delta_i}}\right| \le \frac{\xi_n}{\sigma_{i-1}\sqrt{\Delta_i}}\right)\\
&\quad + \left(\frac{\sqrt{n}}{\xi_n}\right)\frac{1}{T^2}\sum_{i=1}^n\frac{4\xi_n^2\Delta_i}{2\pi\sigma_{i-1}^2}
\end{aligned}$$

$$
\begin{aligned}
& -\left(\frac{\sqrt{n}}{\xi_n}\right)\frac{1}{T^2}\sum_{i=1}^{n}(t_i-t_{i-1})\frac{4\xi_n\Delta_i^{1/2}}{\sqrt{2\pi\sigma_{i-1}^2}}P_{i-1}\left(\left|\frac{Y_i-Y_{i-1}}{\sigma_{i-1}\sqrt{\Delta_i}}\right|\leq\frac{\xi_n}{\sigma_{i-1}\sqrt{\Delta_i}}\right)\\
: \quad & = A_n+B_n+C_n.
\end{aligned}
$$

We have

$$
\begin{aligned}
A_n &= \left(\frac{\sqrt{n}}{\xi_n}\right)\frac{1}{T^2}\sum_{i=1}^{n}(t_i-t_{i-1})^2P_{i-1}\left(\left|\frac{Y_i-Y_{i-1}}{\sigma_{i-1}\sqrt{\Delta_i}}\right|\leq\frac{\xi_n}{\sigma_{i-1}\sqrt{\Delta_i}}\right)\\
&= \left(\frac{\sqrt{n}}{\xi_n}\right)\frac{1}{T^2}\sum_{i=1}^{n}\Delta_i^2\left[\sqrt{\frac{2}{\pi}}\frac{\xi_n}{\sigma_{i-1}}\Delta_i^{-1/2}+O_p\left(\xi_n^3\,\Delta_i^{-3/2}\right)\right]\\
&= \frac{n^{1/2}}{T^2}\sum_{i=1}^{n}\sqrt{\frac{2}{\pi}}\frac{\Delta_i^{3/2}}{\sigma_{i-1}}+\frac{n^{1/2}}{\xi_n}\frac{1}{T^2}\,n\,O_p\left(\xi_n^3\,n^{-1/2}\right)\\
&\overset{p}{\sim} \sqrt{\frac{2}{\pi}}T^{-\frac{3}{2}}\int_0^T\frac{1}{\sigma_s}H_{3/2}^{'}(s)ds+O_p\left(\left(\xi_n\,n^{1/2}\right)^2\right)\\
&\overset{p}{\sim} \sqrt{\frac{2}{\pi}}T^{-\frac{3}{2}}\int_0^T\frac{1}{\sigma_s}H_{3/2}^{'}(s)ds.
\end{aligned}
$$

Also,

$$
\begin{aligned}
B_n &= \frac{n^{1/2}}{\xi_n}\frac{1}{T^2}\frac{2}{\pi}\sum_i\frac{\xi_n^2\,\Delta_i}{\sigma_{i-1}^2}\\
&\overset{p}{\sim} \frac{\xi_n\,n^{1/2}}{T^2}\frac{2}{\pi}\int_0^T\frac{1}{\sigma_s^2}H_1^{'}(s)\;ds=O_p\left(\xi_n\,n^{1/2}\right)\to 0,
\end{aligned}
$$

and

$$
\begin{aligned}
C_n &= -2\,\frac{n^{1/2}}{\xi_n}\frac{1}{T^2}\sum_{i=1}^{n}\Delta_i\left[\sqrt{\frac{2}{\pi}}\frac{\xi_n}{\sigma_{i-1}}\Delta_i^{-1/2}+O_p\left(\xi_n^3\,\Delta_i^{-3/2}\right)\right]\sqrt{\frac{2}{\pi}}\frac{\xi_n\sqrt{\Delta_i}}{\sigma_{i-1}}\\
&\overset{p}{\sim} \frac{n^{1/2}}{\xi_n}\frac{1}{T^2}\sum_{i=1}^{n}\Delta_i\left[\frac{2}{\pi}\frac{\xi_n^2}{\sigma_{i-1}^2}+\sqrt{\frac{2}{\pi}}\frac{1}{\sigma_{i-1}}O_p\left(\xi_n^4\,\Delta_i^{-1}\right)\right]\\
&= \frac{n^{1/2}}{\xi_n}\frac{1}{T^2}\xi_n^2\sum_{i=1}^{n}\left[\frac{2}{\pi}\frac{\Delta_i}{\sigma_{i-1}^2}+\sqrt{\frac{2}{\pi}}\frac{\Delta_i}{\sigma_{i-1}}O_p\left(\xi_n^2\right)\right]\\
&\overset{p}{\sim} \frac{\xi_n\,n^{1/2}}{T^2}\left(\frac{2}{\pi}\int_0^T\frac{1}{\sigma_s^2}H_1^{'}(s)\;ds+O_p\left(\xi_n^2\right)\sqrt{\frac{2}{\pi}}\int_0^T\frac{1}{\sigma_s}H_1^{'}(s)\;ds\right)=O_p\left(\xi_n\,n^{1/2}\right)\to 0
\end{aligned}
$$

since $n^{1/2}\xi_n\to 0$. If $\eta_i\neq 0$, now write

$$
\begin{aligned}
\left(\frac{n}{\xi_n}\right)\sum_{i=1}^{n}E_{i-1}\left[\left(\widetilde{\phi}_{n,i}^{\prime}\right)^2\right] &= \left(\frac{n}{\xi_n}\right)\frac{1}{T^2}\sum_{i=1}^{n}(t_i-t_{i-1})^2P_{i-1}\left(\left|\frac{Y_i-Y_{i-1}}{\sigma_{i-1}\sqrt{\Delta_i}}\right|\leq\frac{\xi_n}{\sigma_{i-1}\sqrt{\Delta_i}}\right)\\
&\quad +\left(\frac{n}{\xi_n}\right)\frac{1}{T^2}\sum_{i=1}^{n}\frac{4\xi_n^2\Delta_i}{\sigma^2(\Delta_i)2\pi\sigma_{i-1}^2}\\
&\quad -\left(\frac{n}{\xi_n}\right)\frac{1}{T^2}\sum_{i=1}^{n}(t_i-t_{i-1})\frac{4\xi_n\Delta_i^{1/2}}{\sigma(\Delta_i)\sqrt{2\pi\sigma_{i-1}^2}}P_{i-1}\left(\left|\frac{Y_i-Y_{i-1}}{\sigma_{i-1}\sqrt{\Delta_i}}\right|\leq\frac{\xi_n}{\sigma_{i-1}\sqrt{\Delta_i}}\right)\\
&: \quad = A_n^*+B_n^*+C_n^*.
\end{aligned}
$$

Using Lemma 3,

$$\begin{aligned}
A_n^* &= \left(\frac{n}{\xi_n}\right)\frac{1}{T^2}\sum_{i=1}^{n}(t_i-t_{i-1})^2 P_{i-1}\left(\left|\frac{Y_i-Y_{i-1}}{\sigma_{i-1}\sqrt{\Delta_i}}\right|\le\frac{\xi_n}{\sigma_{i-1}\sqrt{\Delta_i}}\right)\\
&= \left(\frac{n}{\xi_n}\right)\frac{1}{T^2}\sum_{i=1}^{n}\Delta_i^2\left[\frac{2\,\xi_n}{\sqrt{2\,\pi\,\sigma_\eta^2}}\,e^{-\frac{1}{2}\left(\frac{\eta_{i-1}}{\sigma_\eta}\right)^2}+O_p\left(\xi_n^3+\xi_n\Delta_i\right)\right]\\
&\overset{p}{\sim} \frac{n}{T^2}\sqrt{\frac{2}{\pi}}\,\frac{1}{\sigma_\epsilon}\sum_{i=1}^{n}\Delta_i^2\\
&\overset{p}{\sim} \frac{n}{T^2}\sqrt{\frac{2}{\pi}}\,\frac{1}{\sigma_\epsilon}\,\frac{T}{n}\int_0^T H_2^{'}(s)\;ds\\
&= \frac{1}{T}\sqrt{\frac{2}{\pi}}\,\frac{1}{\sigma_\epsilon}\int_0^T H_2^{'}(s)\;ds\to\sqrt{\frac{2}{\pi}}\,T^{-1}\,\frac{1}{\sigma_\epsilon}\int_0^T H_2^{'}(s)\;ds.
\end{aligned}$$

Now write

$$B_n^*=\frac{n}{\xi_n}\,T^{-2}\sum_{i=1}^{n}\frac{2}{\pi}\,\frac{\xi_n^2\,\Delta_i^2}{\sigma_\epsilon^2}\sim\frac{2}{\pi}\,T^{-1}\,\frac{\xi_n}{\sigma_\epsilon^2}\int_0^T H_2^{'}(s)\;ds=O_p\left(\xi_n\right)\to 0,$$

and

$$\begin{aligned}
C_n^* &= \left(\frac{n}{\xi_n}\right)\frac{1}{T^2}\sum_{i=1}^{n}(t_i-t_{i-1})\frac{4\xi_n\Delta_i^{1/2}}{\sigma(\Delta_i)\sqrt{2\pi\sigma_{i-1}^2}}\left[\frac{2\,\xi_n}{\sqrt{2\,\pi\,\sigma_\eta^2}}\,e^{-\frac{1}{2}\left(\frac{\eta_{i-1}}{\sigma_\eta}\right)^2}+O_p\left(\xi_n^3+\xi_n\Delta_i\right)\right]\\
&\overset{p}{\sim} \frac{n}{T^2}\sum_{i=1}^{n}\frac{2}{\pi}\,\frac{\xi_n}{\sigma_\epsilon^2}\,\Delta_i^2+\frac{n}{T^2}\,O_p\left(\xi_n^3\right)\sum_{i=1}^{n}\Delta_i^2\sqrt{\frac{2}{\pi}}\,\frac{1}{\sigma_\epsilon}\\
&\overset{p}{\sim} \frac{2}{\pi}\,T^{-1}\,\frac{1}{\sigma_\epsilon^2}\,\xi_n\int_0^T H_2^{'}(s)\;ds+O_p\left(\xi_n^2\right)\sqrt{\frac{2}{\pi}}\,T^{-1}\,\frac{1}{\sigma_\epsilon}\int_0^T H_2^{'}(s)\;ds=O_p\left(\xi_n\right)\to 0.
\end{aligned}$$

We now deal with the fourth moment. If $\eta_i=0$,

$$\begin{aligned}
\left(\frac{\sqrt{n}}{\xi_n}\right)^2\sum_{i=1}^{n}E_{i-1}\left[\left(\widetilde{\phi}_{n,i}\right)^4\right] &= \frac{n}{\xi_n^2}\,\frac{1}{T^4}\sum_{i=1}^{n}\Bigg(\Delta_i^4\,\psi_i+\left(\frac{2}{\pi}\right)^2\frac{\xi_n^4\,\Delta_i^2}{\sigma_{i-1}^4}-4\,\Delta_i\,\psi_i\left(\frac{2}{\pi}\right)^{3/2}\frac{\xi_n^3\,\Delta_i^{3/2}}{\sigma_{i-1}^3}\\
&- 4\,\Delta_i^3\,\psi_i\sqrt{\frac{2}{\pi}}\,\frac{\xi_n\,\Delta_i^{1/2}}{\sigma_{i-1}}+6\,\Delta_i^2\,\psi_i\,\frac{2}{\pi}\,\frac{\xi_n^2\,\Delta_i}{\sigma_{i-1}^2}\Bigg)=A_n+B_n+C_n+D_n+E_n.
\end{aligned}$$

Now,

$$\begin{aligned}
A_n &= \frac{n}{\xi_n^2}\,\frac{1}{T^4}\sum_{i=1}^{n}\Delta_i^4\left(\sqrt{\frac{2}{\pi}}\,\frac{\xi_n\,\Delta_i^{-1/2}}{\sigma_{i-1}}+O_p\left(\xi_n^3\,\Delta_i^{-3/2}\right)\right)\\
&= \frac{n}{\xi_n}\,\frac{1}{T^4}\sqrt{\frac{2}{\pi}}\sum_{i=1}^{n}\frac{\Delta_i^{7/2}}{\sigma_{i-1}}+\frac{n}{T^4}\sum_{i=1}^{n}\Delta_i^{5/2}O_p\left(\xi_n\right)\\
&\overset{p}{\sim} \frac{n}{\xi_n}\,\frac{1}{T^4}\sqrt{\frac{2}{\pi}}\,\frac{T^{5/2}}{n^{5/2}}\int_0^T\frac{1}{\sigma_s}\,H_{7/2}^{'}(s)\;ds+O_p\left(\xi_n\right)\frac{n}{T^4}\,\frac{T^{3/2}}{n^{3/2}}\int_0^T H_{5/2}^{'}(s)\;ds\\
&= \frac{1}{\xi_n\,n^{3/2}}\,T^{-3/2}\sqrt{\frac{2}{\pi}}\int_0^T\frac{1}{\sigma_s}\,H_{7/2}^{'}(s)\;ds+\frac{O_p\left(\xi_n\right)}{n^{1/2}}\,T^{-5/2}\int_0^T H_{5/2}^{'}(s)\;ds=O_p\left(\frac{1}{\xi_n\,n^{3/2}}\right)\to 0
\end{aligned}$$

if $\xi_n\, n^{3/2} \to \infty$. Also,

$$\begin{aligned}
B_n &= \frac{n}{\xi_n^2}\left(\frac{2}{\pi}\right)^2 \frac{1}{T^4} \sum_{i=1}^{n} \frac{\xi_n^4\,\Delta_i^2}{\sigma_{i-1}^4} \\
&= \xi_n^2\,\frac{n}{T^4}\left(\frac{2}{\pi}\right)^2 \sum_{i=1}^{n} \frac{\Delta_i^2}{\sigma_{i-1}^4} \\
&\overset{p}{\sim} \xi_n^2\,\frac{n}{T^4}\left(\frac{2}{\pi}\right)^2 \frac{T}{n}\int_0^T \frac{1}{\sigma_s^4}\,H_2^{'}(s)\;ds \\
&= \xi_n^2\,\frac{1}{T^3}\left(\frac{2}{\pi}\right)^2 \int_0^T \frac{1}{\sigma_s^4}\,H_2^{'}(s)\;ds = O_p\left(\xi_n^2\right) \to 0,
\end{aligned}$$

$$\begin{aligned}
C_n &= -4\,\frac{n}{\xi_n^2}\,\frac{1}{T^4}\sum_{i=1}^{n}\Delta_i\,\psi_i\left(\frac{2}{\pi}\right)^{3/2}\frac{\xi_n^3\,\Delta_i^{3/2}}{\sigma_{i-1}^3} \\
&= \frac{n}{\xi_n^2}\,\frac{1}{T^4}\sum_{i=1}^{n}\Delta_i\left[\sqrt{\frac{2}{\pi}}\,\frac{\xi_n\,\Delta_i^{1/2}}{\sigma_{i-1}} + O_p\left(\xi_n^3\,\Delta_i^{-3/2}\right)\right]\left(\frac{2}{\pi}\right)^{3/2}\frac{\xi_n^3\,\Delta_i^{3/2}}{\sigma_{i-1}^3} \\
&\overset{p}{\sim} n\,\xi_n^2\,T^{-4}\left(\frac{2}{\pi}\right)^{5/2}\sum_{i=1}^{n}\frac{\Delta_i^2}{\sigma_{i-1}^4} + n\,\xi_n^4\,T^{-4}\left(\frac{2}{\pi}\right)^{3/2}\sum_{i=1}^{n}\frac{\Delta_i}{\sigma_{i-1}^3} \\
&\overset{p}{\sim} \xi_n^2\,T^{-3}\left(\frac{2}{\pi}\right)^{5/2}\int_0^T\frac{1}{\sigma_s^4}\,H_2^{'}(s)\;ds + n\,\xi_n^4\,T^{-4}\left(\frac{2}{\pi}\right)^{3/2}\int_0^T\frac{1}{\sigma_s^3}\,H_1^{'}(s)\;ds = O_p\left(\xi_n^2\right) \to 0,
\end{aligned}$$

and

$$\begin{aligned}
D_n &= -4\,\frac{n}{\xi_n^2}\,\frac{1}{T^4}\sum_{i=1}^{n}\Delta_i^3\,\psi_i\sqrt{\frac{2}{\pi}}\,\frac{\xi_n\,\Delta_i^{1/2}}{\sigma_{i-1}} \\
&= \frac{n}{\xi_n^2}\,\frac{1}{T^4}\sum_{i=1}^{n}\Delta_i^3\left[\sqrt{\frac{2}{\pi}}\,\frac{\xi_n\,\Delta_i^{-1/2}}{\sigma_{i-1}} + O_p\left(\xi_n^3\,\Delta_i^{-3/2}\right)\right]\sqrt{\frac{2}{\pi}}\,\frac{\xi_n\,\Delta_i^{1/2}}{\sigma_{i-1}} \\
&\overset{p}{\sim} n\,T^{-4}\,\frac{2}{\pi}\sum_{i=1}^{n}\frac{\Delta_i^3}{\sigma_{i-1}^2} + n\,\xi_n^2\,T^{-4}\sqrt{\frac{2}{\pi}}\sum_{i=1}^{n}\frac{\Delta_i^2}{\sigma_{i-1}} \\
&\overset{p}{\sim} \frac{1}{n}\,T^{-2}\,\frac{2}{\pi}\int_0^T\frac{1}{\sigma_s^2}\,H_3^{'}(s)\;ds + \xi_n^2\,T^{-3}\sqrt{\frac{2}{\pi}}\int_0^T\frac{1}{\sigma_s}\,H_2^{'}(s)\;ds = O_p\left(\frac{1}{n}\right) \to 0.
\end{aligned}$$

Finally,

$$\begin{aligned}
E_n &= 6\,\frac{n}{\xi_n^2}\,\frac{1}{T^4}\sum_{i=1}^{n}\Delta_i^2\,\psi_i\,\frac{2}{\pi}\,\frac{\xi_n^2\,\Delta_i}{\sigma_{i-1}^2} \\
&= 6\,\frac{n}{\xi_n^2}\,\frac{1}{T^4}\sum_{i=1}^{n}\Delta_i^2\left[\sqrt{\frac{2}{\pi}}\,\frac{\xi_n\,\Delta_i^{-1/2}}{\sigma_{i-1}} + O_p\left(\xi_n^3\,\Delta_i^{-3/2}\right)\right]\frac{2}{\pi}\,\frac{\xi_n^2\,\Delta_i}{\sigma_{i-1}^2} \\
&\overset{p}{\sim} n\,\xi_n\,\frac{1}{T^4}\left(\frac{2}{\pi}\right)^{3/2}\sum_{i=1}^{n}\frac{\Delta_i^{5/2}}{\sigma_{i-1}^3} + n\,\xi_n^3\,\frac{1}{T^4}\left(\frac{2}{\pi}\right)\sum_{i=1}^{n}\frac{\Delta_i^{3/2}}{\sigma_{i-1}^2} \\
&\overset{p}{\sim} n\,\xi_n\,\frac{1}{T^4}\left(\frac{2}{\pi}\right)^{3/2}\left(\frac{T}{n}\right)^{3/2}\int_0^T\frac{1}{\sigma_s^3}\,H_{5/2}^{'}(s)\;ds + n\,\xi_n^3\,\frac{1}{T^4}\left(\frac{2}{\pi}\right)\left(\frac{T}{n}\right)^{1/2}\int_0^T\frac{1}{\sigma_s^2}\,H_{3/2}^{'}(s)\;ds \\
&= \frac{\xi_n}{\sqrt{n}}\,T^{-5/2}\left(\frac{2}{\pi}\right)^{3/2}\int_0^T\frac{1}{\sigma_s^3}\,H_{5/2}^{'}(s)\;ds + \xi_n^3\,\sqrt{n}\,T^{-7/2}\,\frac{2}{\pi}\int_0^T\frac{1}{\sigma_s^2}\,H_{5/2}^{'}(s)\;ds = O_p\left(\frac{\xi_n}{\sqrt{n}}\right) \to 0.
\end{aligned}$$

Instead, if $\eta_i \neq 0$,

$$\left(\frac{n}{\xi_n}\right)^2 \sum_{i=1}^n E_{i-1}\left[\left(\widetilde{\phi}'_{n,i}\right)^4\right] = \left(\frac{n}{\xi_n}\right)^2 \frac{1}{T^4} \sum_{i=1}^n \left(\Delta_i^4\,\psi_i + \left(\frac{2}{\pi}\right)^2 \frac{\xi_n^4\,\Delta_i^4}{\sigma_\epsilon^4} - 4\,\Delta_i\,\psi_i\,\left(\frac{2}{\pi}\right)^{3/2} \frac{\xi_n^3\,\Delta_i^3}{\sigma_\epsilon^3}\right.$$
$$\left. - \; 4\,\Delta_i^3\,\psi_i \sqrt{\frac{2}{\pi}}\,\frac{\xi_n\,\Delta_i}{\sigma_\epsilon} + 6\,\Delta_i^2\,\psi_i\,\frac{2}{\pi}\,\frac{\xi_n^2\,\Delta_i^2}{\sigma_\epsilon^2}\right) = A_n + B_n + C_n + D_n + E_n.$$

We have

$$\begin{aligned}
A_n &= \left(\frac{n}{\xi_n}\right)^2 \frac{1}{T^4} \sum_{i=1}^n \Delta_i^4\,\psi_i = \left(\frac{n}{\xi_n}\right)^2 \frac{1}{T^4} \sum_{i=1}^n \Delta_i^4 \left[\frac{2\,\xi_n}{\sqrt{2\,\pi\,\sigma_\eta^2}}\,e^{-\frac{1}{2}\left(\frac{\eta_{i-1}}{\sigma_\eta}\right)^2} + O_p\left(\xi_n^3 + \xi_n\Delta_i\right)\right] \\
&\overset{p}{\sim} \frac{n^2}{\xi_n}\,\frac{1}{T^4}\sqrt{\frac{2}{\pi}}\,\frac{1}{\sigma_\epsilon}\,\frac{T^3}{n^3}\int_0^T H_4^{'}(s)\;ds + O_p(\xi_n)\;n^2\,\frac{1}{T^4}\,\frac{T^3}{n^3}\int_0^T H_4^{'}(s)\;ds \\
&= \frac{1}{\xi_n\,n}\,T^{-1}\sqrt{\frac{2}{\pi}}\,\frac{1}{\sigma_\epsilon}\int_0^T H_4^{'}(s)\;ds + O_p\left(\frac{\xi_n}{n}\right)\,T^{-1}\int_0^T H_4^{'}(s)\;ds = O_p\left(\frac{1}{\xi_n\,n}\right) \to 0
\end{aligned}$$

if $\xi_n\,n \to \infty$. Now,

$$\begin{aligned}
B_n &= \left(\frac{n}{\xi_n}\right)^2 \frac{1}{T^4}\left(\frac{2}{\pi}\right)^2 \sum_{i=1}^n \frac{\xi_n^4\,\Delta_i^4}{\sigma_\epsilon^4} = n^2\,\xi_n^2\,\frac{1}{T^4}\left(\frac{2}{\pi}\right)^2 \frac{1}{\sigma_\epsilon^4}\sum_{i=1}^n \Delta_i^4 \\
&\overset{p}{\sim} n^2\,\xi_n^2\,\frac{1}{T^4}\left(\frac{2}{\pi}\right)^2 \frac{1}{\sigma_\epsilon^4}\,\frac{T^3}{n^3}\int_0^T H_4^{'}(s) = \frac{\xi_n^2}{n}\,T^{-1}\left(\frac{2}{\pi}\right)^2 \frac{1}{\sigma_\epsilon^4}\int_0^T H_4^{'}(s) = O_p\left(\frac{\xi_n^2}{n}\right) \to 0.
\end{aligned}$$

Write,

$$\begin{aligned}
C_n &= -4\left(\frac{n}{\xi_n}\right)^2 \frac{1}{T^4}\sum_{i=1}^n \Delta_i\,\psi_i\left(\frac{2}{\pi}\right)^{3/2}\frac{\xi_n^3\,\Delta_i^3}{\sigma_\epsilon^3} \\
&= \left(\frac{n}{\xi_n}\right)^2 \frac{1}{T^4}\sum_{i=1}^n \Delta_i\left[\frac{2\,\xi_n}{\sqrt{2\,\pi\,\sigma_\eta^2}}\,e^{-\frac{1}{2}\left(\frac{\eta_{i-1}}{\sigma_\eta}\right)^2} + O_p\left(\xi_n^3 + \xi_n\Delta_i\right)\right]\left(\frac{2}{\pi}\right)^{3/2}\frac{\xi_n^3\,\Delta_i^3}{\sigma_\epsilon^3} \\
&\overset{p}{\sim} \left(\frac{n}{\xi_n}\right)^2 \frac{1}{T^4}\sum_{i=1}^n \Delta_i\sqrt{\frac{2}{\pi}}\,\frac{\xi_n}{\sigma_\epsilon}\left(\frac{2}{\pi}\right)^{3/2}\frac{\xi_n^3\,\Delta_i^3}{\sigma_\epsilon^3} + \left(\frac{n}{\xi_n}\right)^2 \frac{1}{T^4}\sum_{i=1}^n \Delta_i\,\xi_n^3\left(\frac{2}{\pi}\right)^{3/2}\frac{\xi_n^3\,\Delta_i^3}{\sigma_\epsilon^3} \\
&= n^2\,\xi_n^2\,\frac{1}{T^4}\left(\frac{2}{\pi}\right)^2\frac{1}{\sigma_\epsilon^4}\sum_{i=1}^n \Delta_i^4 + n^2\,\xi_n^4\,\frac{1}{T^4}\left(\frac{2}{\pi}\right)^{3/2}\frac{1}{\sigma_\epsilon^3}\sum_{i=1}^4 \Delta_i^4 \\
&\overset{p}{\sim} n^2\,\xi_n^2\,\frac{1}{T^4}\left(\frac{2}{\pi}\right)^2\frac{1}{\sigma_\epsilon^4}\,\frac{T^3}{n^3}\int_0^T H_4^{'}(s)\;ds + n^2\,\xi_n^4\,\frac{1}{T^4}\left(\frac{2}{\pi}\right)^{3/2}\frac{1}{\sigma_\epsilon^3}\,\frac{T^3}{n^3}\int_0^T H_4^{'}(s)\;ds \\
&= \frac{\xi_n^2}{n}\,\frac{1}{T}\left(\frac{2}{\pi}\right)^2\frac{1}{\sigma_\epsilon^4}\int_0^T H_4^{'}(s)\;ds + \frac{\xi_n^4}{n}\,\frac{1}{T}\left(\frac{2}{\pi}\right)^{3/2}\frac{1}{\sigma_\epsilon^3}\int_0^T H_4^{'}(s)\;ds = O_p\left(\frac{\xi_n^2}{n}\right) \to 0,
\end{aligned}$$

and

$$\begin{aligned}
D_n &= -4\left(\frac{n}{\xi_n}\right)^2 \frac{1}{T^4}\sum_{i=1}^n \Delta_i^3\,\psi_i\sqrt{\frac{2}{\pi}}\,\frac{\xi_n\,\Delta_i}{\sigma_\epsilon} \\
&= \left(\frac{n}{\xi_n}\right)^2 \frac{1}{T^4}\sum_{i=1}^n \Delta_i^3\left[\frac{2\,\xi_n}{\sqrt{2\,\pi\,\sigma_\eta^2}}\,e^{-\frac{1}{2}\left(\frac{\eta_{i-1}}{\sigma_\eta}\right)^2} + O_p\left(\xi_n^3 + \xi_n\Delta_i\right)\right]\sqrt{\frac{2}{\pi}}\,\frac{\xi_n\,\Delta_i}{\sigma_\epsilon}
\end{aligned}$$

$$
\begin{aligned}
&\overset{p}{\sim} \left(\frac{n}{\xi_n}\right)^2 \frac{1}{T^4}\frac{2}{\pi}\frac{\xi_n^2}{\sigma_\epsilon^2}\sum_{i=1}^n \Delta_i^4 + \left(\frac{n}{\xi_n}\right)^2 \frac{1}{T^4}\sqrt{\frac{2}{\pi}}\frac{\xi_n}{\sigma_\epsilon}\sum_{i=1}^n \Delta_i^4\, O_p\left(\xi_n^3 + \xi_n \Delta_i\right) \\
&\overset{p}{\sim} n^2 \frac{1}{T^4}\frac{2}{\pi}\frac{1}{\sigma_\epsilon^2}\frac{T^3}{n^3}\int_0^T H_4^{'}(s)\, ds + \xi_n^2 n^2 \frac{1}{T^4}\sqrt{\frac{2}{\pi}}\frac{1}{\sigma_\epsilon}\frac{T^3}{n^3}\int_0^T H_4^{'}(s)\, ds \\
&= \frac{1}{n}T^{-1}\frac{2}{\pi}\frac{1}{\sigma_\epsilon^2}\int_0^T H_4^{'}(s)\, ds + \frac{\xi_n^2}{n}T^{-1}\sqrt{\frac{2}{\pi}}\frac{1}{\sigma_\epsilon}\int_0^T H_4^{'}(s)\, ds = O_p\left(\frac{1}{n}\right) \to 0.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
E_n &= 6\left(\frac{n}{\xi_n}\right)^2 \frac{1}{T^4}\sum_{i=1}^n \Delta_i^2\, \psi_i \frac{2}{\pi}\frac{\xi_n^2\, \Delta_i^2}{\sigma_\epsilon^2} \\
&= \left(\frac{n}{\xi_n}\right)^2 \frac{1}{T^4}\sum_{i=1}^n \Delta_i^2 \left[\frac{2\,\xi_n}{\sqrt{2\,\pi\,\sigma_\eta^2}}\, e^{-\frac{1}{2}\left(\frac{\eta_{i-1}}{\sigma_\eta}\right)^2} + O_p\left(\xi_n^3 + \xi_n\Delta_i\right)\right]\frac{2}{\pi}\frac{\xi_n^2\,\Delta_i^2}{\sigma_\epsilon^2} \\
&\overset{p}{\sim} \left(\frac{n}{\xi_n}\right)^2 \frac{1}{T^4}\sqrt{\frac{2}{\pi}}\frac{\xi_n}{\sigma_\epsilon}\frac{2}{\pi}\frac{\xi_n^2}{\sigma_\epsilon^2}\sum_{i=1}^n \Delta_i^4 + \left(\frac{n}{\xi_n}\right)^2 \frac{1}{T^4}\sum_{i=1}^n \Delta_i^4\, O_p\left(\xi_n^3\right)\frac{2}{\pi}\frac{\xi_n^2}{\sigma_\epsilon^2} \\
&\overset{p}{\sim} \xi_n n^2 \frac{1}{T^4}\left(\frac{2}{\pi}\right)^{3/2}\frac{1}{\sigma_\epsilon^3}\frac{T^3}{n^3}\int_0^T H_4^{'}(s)\, ds + \xi_n^3 n^2 \frac{1}{T^4}\frac{2}{\pi}\frac{1}{\sigma_\epsilon^2}\frac{T^3}{n^3}\int_0^T H_4^{'}(s)\, ds \\
&= \frac{\xi_n}{n}T^{-1}\left(\frac{2}{\pi}\right)^{3/2}\frac{1}{\sigma_\epsilon^3}\int_0^T H_4^{'}(s)\, ds + \frac{\xi_n^3}{n}T^{-1}\frac{2}{\pi}\frac{1}{\sigma_\epsilon^3}\int_0^T H_4^{'}(s)\, ds = O_p\left(\frac{\xi_n}{n}\right) \to 0.
\end{aligned}
$$

We now turn to stable convergence. Denote by $Y_i^{(n)} - Y_{i-1}^{(n)}$ the discretized process without drift. Write

$$
\begin{aligned}
&\left(\frac{\sqrt{n}}{\xi_n}\right)^{1/2}\sum_{i=1}^n E_{i-1}\left[\widetilde{\phi}_{n,i}^{(n)}(W_{t_i} - W_{t_{i-1}})\right] \\
&= \left(\frac{\sqrt{n}}{\xi_n}\right)^{1/2}\frac{1}{T}\sum_{i=1}^n (t_i - t_{i-1})E_{i-1}\left[1_{\left\{\left|\frac{Y_i^{(n)}-Y_{i-1}^{(n)}}{\sigma_{i-1}\sqrt{\Delta_i}}\right| \le \frac{\xi_n}{\sigma_{i-1}\sqrt{\Delta_i}}\right\}}(W_{t_i} - W_{t_{i-1}})\right] \\
&\le \left(\frac{\sqrt{n}}{\xi_n}\right)^{1/2}\frac{1}{T}\sum_{i=1}^n (t_i - t_{i-1})\left(E_{i-1}\left[1_{\left\{\left|\frac{Y_i^{(n)}-Y_{i-1}^{(n)}}{\sigma_{i-1}\sqrt{\Delta_i}}\right| \le \frac{\xi_n}{\sigma_{i-1}\sqrt{\Delta_i}}\right\}}\right]\right)^{1/p}\left(E_{i-1}\left[(W_{t_i} - W_{t_{i-1}})^{\frac{p}{p-1}}\right]\right)^{\frac{p-1}{p}} \\
&\quad \text{(by Holder's inequality)} \\
&\le C_p\left(\frac{\sqrt{n}}{\xi_n}\right)^{1/2}\frac{1}{T}\sum_{i=1}^n (t_i - t_{i-1})\left(E_{i-1}\left[1_{\left\{\left|\frac{Y_i^{(n)}-Y_{i-1}^{(n)}}{\sigma_{i-1}\sqrt{\Delta_i}}\right| \le \frac{\xi_n}{\sigma_{i-1}\sqrt{\Delta_i}}\right\}}\right]\right)^{1/p}\left(\left[\Delta_i{}^{\frac{p}{2(p-1)}}\right]\right)^{(p-1)/p} \\
&\quad \text{(by BDS inequality)} \\
&= O_p(n^{1/4}\xi_n^{-1/2} n n^{-1}\xi_n^{1/p} n^{1/(2p)} n^{-1/2}) \\
&= O_p(n^{1/(2p)-1/4}\xi_n^{1/p-1/2}). \qquad \text{(A.5)}
\end{aligned}
$$

If $p \to 1$, the bound goes to $n^{1/4}\xi_n^{1/2}$ but $n^{1/4}\xi_n^{1/2} \to 0$ since $n^{7/10}\xi_n \to 0$. Now consider an orthogonal martingale N_t. Write

$$
\left(\frac{\sqrt{n}}{\xi_n}\right)^{1/2}\sum_{i=1}^n E_{i-1}\left[\widetilde{\phi}_{n,i}^{*}(N_{t_i} - N_{t_{i-1}})\right]
$$

$$= \left(\frac{\sqrt{n}}{\xi_n}\right)^{1/2} \frac{1}{T}\sum_{i=1}^{n}(t_i - t_{i-1})E_{i-1}\left[1_{\left\{\left|\frac{Y_i^{(n)}-Y_{i-1}^{(n)}}{\sigma_{i-1}\sqrt{\Delta_i}}\right|\leq\frac{\xi_n}{\sigma_{i-1}\sqrt{\Delta_i}}\right\}}(N_{t_i} - N_{t_{i-1}})\right].$$

Using a martingale representation for $1_{\left\{\left|\frac{Y_i^{(n)}-Y_{i-1}^{(n)}}{\sigma_{i-1}\sqrt{\Delta_i}}\right|\leq\frac{\xi_n}{\sigma_{i-1}\sqrt{\Delta_i}}\right\}}$, which is an an L^2-function, it is clear that the term is zero since $[W,N]=0$. Hence, stable convergence holds for the discretized process without drift (under P_n^*). We show that it is also true for the non-discretized process without drift (i.e., the stochastic integral) by applying Theorem 2 of Mykland and Zhang (2009). To this extent, consider the martingale

$$M_n(0) = \frac{1}{12}\sum_{i=0}^{n}(t_i - t_{i-1})^{1/2}k_{t_{i-1}}h\left(\frac{W_{t_i} - W_{t_{i-1}}}{(t_i - t_{i-1})^{1/2}}\right),$$

where $h(.)$ is a third order Hermite polynomial and $k_{t_{i-1}}$ is measurable with respect to t_{i-1} information. Call $A_{1,2}$, the quadratic co-variation between $\left(\frac{\sqrt{n}}{\xi_n}\right)^{1/2}\sum_{i=1}^{n}\widetilde{\phi}_{n,i}$ and $M_n(0)$ and write

$$\begin{aligned}
A_{1,2} &= \frac{1}{12}\left(\frac{\sqrt{n}}{\xi_n}\right)^{1/2}\sum_{i=1}^{n}E_{i-1}\left[\widetilde{\phi}_{n,i}\left((t_i - t_{i-1})^{1/2}k_{t_{i-1}}h\left(\frac{W_{t_i} - W_{t_{i-1}}}{(t_i - t_{i-1})^{1/2}}\right)\right)\right] \\
&= \frac{1}{12}\left(\frac{\sqrt{n}}{\xi_n}\right)^{1/2}\sum_{i=1}^{n}(t_i - t_{i-1})^{3/2}k_{t_{i-1}}E_{i-1}\left[1_{\left\{Y_i^{(n)}-Y_{i-1}^{(n)}\leq\xi_n\right\}}\left(h\left(\frac{W_{t_i} - W_{t_{i-1}}}{(t_i - t_{i-1})^{1/2}}\right)\right)\right] \\
&= \frac{1}{12}\left(\frac{\sqrt{n}}{\xi_n}\right)^{1/2}\sum_{i=1}^{n}(t_i - t_{i-1})^{3/2}k_{t_{i-1}}E_{i-1}\left[1_{\left\{Y_i^{(n)}-Y_{i-1}^{(n)}\leq\xi_n\right\}}\left(\left(\frac{W_{t_i} - W_{t_{i-1}}}{(t_i - t_{i-1})^{1/2}}\right)^3\right)\right] \\
&\quad -\frac{1}{12}\left(\frac{\sqrt{n}}{\xi_n}\right)^{1/2}\sum_{i=1}^{n}(t_i - t_{i-1})^{3/2}k_{t_{i-1}}E_{i-1}\left[1_{\left\{Y_i^{(n)}-Y_{i-1}^{(n)}\leq\xi_n\right\}}\left(\frac{W_{t_i} - W_{t_{i-1}}}{(t_i - t_{i-1})^{1/2}}\right)\right].
\end{aligned}$$

Now, write

$$\begin{aligned}
&\frac{1}{12}\left(\frac{\sqrt{n}}{\xi_n}\right)^{1/2}\sum_{i=1}^{n}(t_i - t_{i-1})^{3/2}k_{t_{i-1}}E_{i-1}\left[1_{\left\{Y_i^{(n)}-Y_{i-1}^{(n)}\leq\xi_n\right\}}\left(\left(\frac{W_{t_i} - W_{t_{i-1}}}{(t_i - t_{i-1})^{1/2}}\right)^k\right)\right] \\
\leq &\left(\frac{\sqrt{n}}{\xi_n}\right)^{1/2}\frac{1}{T}\sum_{i=1}^{n}(t_i - t_{i-1})^{3/2-k/2}\left(E_{i-1}\left[1_{\left\{\left|Y_i^{(n)}-Y_{i-1}^{(n)}\right|\leq\xi_n\right\}}\right]\right)^{1/p}\left(E_{i-1}\left[(W_{t_i} - W_{t_{i-1}})^{k\frac{p}{p-1}}\right]\right)^{\frac{p-1}{p}} \\
\leq &\; C_p\left(\frac{\sqrt{n}}{\xi_n}\right)^{1/2}\frac{1}{T}\sum_{i=1}^{n}(t_i - t_{i-1})^{3/2-k/2}\left(E_{i-1}\left[1_{\left\{\left|Y_i^{(n)}-Y_{i-1}^{(n)}\right|\leq\xi_n\right\}}\right]\right)^{1/p}\left(\left[\Delta_i^{\frac{kp}{2(p-1)}}\right]\right)^{(p-1)/p} \\
= &\; O_p(n^{1/4}\xi_n^{-1/2}nn^{-3/2+k/2}\xi_n^{1/p}n^{1/(2p)}n^{-k/2}) \\
= &\; O_p(n^{1/4}\xi_n^{-1/2}nn^{-3/2}\xi_n^{1/p}n^{1/(2p)}) \\
= &\; O_p(n^{5/4-3/2+1/(2p)}\xi_n^{1/p-1/2}) \\
= &\; O_p(n^{1/(2p)-1/4}\xi_n^{1/p-1/2}),
\end{aligned}$$

with $k=1,2$. If $p\to 1$, the bound goes to $n^{1/4}\xi_n^{1/2}$ but $n^{1/4}\xi_n^{1/2}\to 0$ since $n^{7/10}\xi_n\to 0$. This proves that $A_{1,2}\xrightarrow{p} 0$ implying that stable convergence holds also for the non-discretized process without drift (under P^*). By Theorem 1 in Mykland and Zhang (2009), the drift can now be put back in and the convergence is stable under the original measure P.

Part 3: the proof under the alternative.

Using Eq. (4.4), we have

$$p_{t_i} - p_{t_{i-1}} = (\widetilde{p}_{t_i} - p_{t_{i-1}})(1 - B_{i,n})$$

so that we can write

$$\begin{aligned}
\mathrm{IT} &= \frac{1}{T}\sum_{i=1}^{n}(t_i - t_{i-1})1_{\{|p_{t_i}-p_{t_{i-1}}|\le\xi_n\}} = \frac{1}{T}\sum_{i=1}^{n}(t_i - t_{i-1})1_{\{|(\widetilde{p}_{t_i}-p_{t_{i-1}})(1-B_{i,n})|\le\xi_n\}} \\
&= \frac{1}{T}\sum_{i=1}^{n}(t_i - t_{i-1})B_{i,n} + \underbrace{\frac{1}{T}\sum_{i=1}^{n}(1-B_{i,n})(t_i - t_{i-1})1_{\{|\widetilde{p}_{t_i}-p_{t_{i-1}}|\le\xi_n\}}}_{\mathrm{IT}_1}.
\end{aligned}$$

The first part of IT converges, in probability, to p^F by Eq. (4.2). For IT_1, we have

$$\mathrm{IT}_1 \le \frac{1}{T}\sum_{i=1}^{n}(t_i - t_{i-1})1_{\{|\widetilde{p}_{t_i}-p_{t_{i-1}}|\le\xi_n\}} := \mathrm{IT}_2,$$

where IT_2 is a positive quantity. In order to prove its convergence to zero in probability, it is enough to establish the convergence to zero of its expected value. To do so, denote by $\overline{K}_i \ge 0$ the number of consecutive flat trades before t_{i-1}. By Assumptions 2 and 3 (or 3′), $\overline{K}_i(t_i - t_{i-1}) \le C_4\frac{K_n}{n} \xrightarrow{p} 0$, thus

$$E\left[1_{\{|\widetilde{p}_{t_i}-p_{t_{i-1}}|\le\xi_n\}}\right] = E\left[1_{\{|\widetilde{p}_{t_i}-\widetilde{p}_{t_{i-\overline{K}_i-1}}|\le\xi_n\}}\right].$$

Now, when $\sigma_\eta^2 > 0$, the last term is $O_p(\xi_n)$, see Eq. (A.3). When $\sigma_\eta^2 = 0$, we have

$$\begin{aligned}
E\left[1_{\{|\widetilde{p}_{t_i}-\widetilde{p}_{t_{i-\overline{K}_i-1}}|\le\xi_n\}}\right] =& E\left[E\left[1_{\{|\widetilde{p}_{t_i}-\widetilde{p}_{t_{i-\overline{K}_i-1}}|\le\xi_n\}}\,|\overline{K}_i\right]\right] \\
=& E\left[\sqrt{\frac{2}{\pi}}\frac{\xi_n}{\sqrt{(t_i - t_{i-1})\left(\overline{K}_i+1\right)}}\frac{1}{\sigma_{t_{i-\overline{K}_i-1}}}\right] \le \sqrt{\frac{2}{\pi}}\frac{1}{C_2}\frac{1}{\sqrt{C_4}}\xi_n n^{1/2}.
\end{aligned}$$

This implies $\mathrm{IT}_1 = o_p(1)$. Now, write

$$\begin{aligned}
\mathrm{EXIT} =& \frac{1}{n}\sum_{i=1}^{n}(t_i - t_{i-1})1_{\left\{|p_{t_i}-p_{t_{i-1}}|\le\xi_n\right\}} - bias \\
=& p^F + \left(\frac{1}{n}\sum_{i=1}^{n}(t_i - t_{i-1})B_{i,n} - p^F\right) + \mathrm{IT}_1 - bias \\
=& p^F + o_p(1),
\end{aligned}$$

where we used $\mathrm{IT}_1 = o_p(1)$, Eq. (4.2) and the fact that the bias is vanishing. This completes the proof.

Proof of Remark 3. Since

$$\begin{aligned}
\widetilde{p}_{t_i} - \widetilde{p}_{t_{i-1}} &= p_{t_i} - p_{t_{i-1}} + \varepsilon_{t_i} \\
&= r_{t_i} + (\phi - 1)(r_{t_i} - r_{t_{i-1}}) + \eta_{t_i} - \eta_{t_{i-1}},
\end{aligned}$$

write

$$Y_\Delta = \frac{p_\Delta + \varepsilon}{(\phi\sigma_0 + (\phi-1)^2\sigma_{-1})\sqrt{\Delta}} = \frac{p_\Delta + \varepsilon}{\widetilde{\sigma}_0\sqrt{\Delta}}.$$

Thus, $\Psi_{Y_\Delta}(t)$ can be factorized as follows

$$\begin{aligned}\Psi_{Y_\Delta}(t) &= \mathrm{E}_0\left[e^{it\frac{p_\Delta+\varepsilon}{\tilde\sigma_0\sqrt{\Delta}}}\right] = \mathrm{E}_0\left[e^{it\frac{X_\Delta-(\phi\mu_0-(\phi-1)\mu_{-1})\Delta}{\tilde\sigma_0\sqrt{\Delta}}} e^{it\frac{(\phi\mu_0-(\phi-1)\mu_{-1})\Delta}{\tilde\sigma_0\sqrt{\Delta}}} e^{it\frac{\varepsilon}{\tilde\sigma_0\sqrt{\Delta}}}\right] \\ &= e^{it\frac{\tilde\mu_0\Delta-\varepsilon}{\tilde\sigma_0\sqrt{\Delta}}}\mathrm{E}_0\left[e^{it\frac{p_\Delta-\tilde\mu_0\Delta}{\tilde\sigma_0\sqrt{\Delta}}}\right]\mathrm{E}_0\left[e^{it\frac{\eta_\Delta}{\sigma_0\sqrt{\Delta}}}\right] = e^{it\frac{\tilde\mu_0\Delta-\varepsilon}{\tilde\sigma_0\sqrt{\Delta}}}\mathrm{E}_0\left[e^{it\frac{p_\Delta-\tilde\mu_0\Delta}{\tilde\sigma_0\sqrt{\Delta}}}\right] e^{-\frac{t^2}{2}\left(\frac{\sigma_\eta^2}{\tilde\sigma_0^2\Delta}\right)},\end{aligned}$$

where $\widetilde{\mu}_0 = (\phi\mu_0 - (\phi-1)\mu_{-1})$. Now, using Bandi and Renò (2013), write

$$\mathrm{E}_0\left[e^{it\frac{p_\Delta-\tilde\mu_0\Delta}{\tilde\sigma_0\sqrt{\Delta}}}\right] = e^{-\frac{t^2}{2}}(1+c\sqrt{\Delta}(it)^3+o(\sqrt{\Delta}))$$

where c is a suitable constant. Finally,

$$\Psi_{Y_\Delta}(t) = e^{it\frac{\tilde\mu_0\Delta-\varepsilon}{\tilde\sigma_0\sqrt{\Delta}}-\frac{t^2}{2}\left(1+\frac{\sigma_\eta^2}{\tilde\sigma_0^2\Delta}\right)}(1+c\sqrt{\Delta}(it)^3+o(\sqrt{\Delta}))$$

and we can proceed as earlier.

Proof of Remark 4. Assume the noise process is stationary in transaction time with $\eta_i = \rho\eta_{i-1} + u_i$, then $\sigma_\eta^2 = \frac{\sigma_u^2}{1-\rho^2}$ is the unconditional variance of the noise. In what follows we will denote $Var(\eta_i|i-1)$ by $\sigma_\eta^2(i-1)$. Clearly, $\sigma_\eta^2(i-1) = \sigma_u^2$ for all i. Now, write

$$\begin{aligned}&\frac{1}{T}\sum_{i=1}^n E_{i-1}\left[(t_i-t_{i-1})1_{\left\{\left|\frac{Y_i-Y_{i-1}}{\sigma_{i-1}\sqrt{\Delta_i}}\right|\le\frac{\xi_n}{\sigma_{i-1}\sqrt{\Delta_i}}\right\}}\right] = \frac{1}{T}\sum_{i=1}^n (t_i-t_{i-1})\frac{2e^{-\frac{1}{2}\left(\frac{\eta_{i-1}}{(\sigma_{i-1}\sqrt{\Delta})\sigma(\Delta)}\right)^2}\xi_n}{\sigma(\Delta_i)\sqrt{2\pi\sigma_{i-1}^2\Delta_i}} \\ &\overset{p}{\sim}\frac{1}{T}\sum_{i=1}^n (t_i-t_{i-1})\frac{2e^{-\frac{1}{2}\left(\frac{\eta_{i-1}}{\sigma_u}\right)^2}\xi_n}{\sqrt{2\pi\sigma_u^2}} \overset{p}{\sim} 2\xi_n\left(T^{-1}\int_0^T H_1'(s)ds\right)\frac{1}{\sqrt{2\pi\sigma_u^2}}E\left(e^{-\frac{1}{2}\left(\frac{X}{\sigma_u}\right)^2}\right) \\ &= 2\xi_n\frac{1}{2\pi\sigma_u\sigma_\eta}\int e^{-\frac{1}{2}\left(\frac{X}{\sigma_u}\right)^2}e^{-\frac{1}{2}\left(\frac{X}{\sigma_\eta}\right)^2}dX = 2\xi_n\frac{\sqrt{1-\rho^2}}{2\pi\sigma_u^2}\int e^{-\frac{1}{2}\left(\frac{X}{\sigma_u}\right)^2}e^{-\frac{1}{2}(1-\rho^2)\left(\frac{X}{\sigma_u}\right)^2}dX \\ &= 2\xi_n\frac{\sqrt{1-\rho^2}}{\sigma_u}\left(\frac{1}{2\pi}\int e^{-\frac{1}{2}(Y)^2}e^{-\frac{1}{2}(1-\rho^2)(Y)^2}dY\right) = 2\xi_n\frac{\sqrt{1-\rho^2}}{\sigma_u}\left(\frac{1}{2\pi}\int e^{-\left(1-\frac{1}{2}\rho^2\right)(Y)^2}dY\right) \\ &= 2\xi_n\frac{\sqrt{1-\rho^2}}{\sigma_u}\frac{\sqrt{1-\frac{1}{2}\rho^2}}{\sqrt{1-\frac{1}{2}\rho^2}}\left(\frac{1}{2\pi}\int e^{-\left(1-\frac{1}{2}\rho^2\right)(Y)^2}dY\right) \\ &= 2\xi_n\frac{1}{\sigma_u}\frac{\sqrt{1-\rho^2}}{\sqrt{1-\frac{1}{2}\rho^2}}\frac{1}{2\sqrt{\pi}}.\end{aligned}$$

Similarly,

$$\frac{1}{T}\sum_{i=1}^n E_{i-1}\left[(t_i-t_{i-1})^2 1_{\left\{\left|\frac{Y_i-Y_{i-1}}{\sigma_{i-1}\sqrt{\Delta_i}}\right|\le\frac{\xi_n}{\sigma_{i-1}\sqrt{\Delta_i}}\right\}}\right] \overset{p}{\sim} (\xi_n/n)\frac{1}{T}\frac{1}{\sqrt{\pi}}\frac{\sqrt{1-\rho^2}}{\sqrt{1-\frac{1}{2}\rho^2}}\frac{H_2(T)}{\sigma_u}.$$

B Appendix: practical implementation

Our implementation of EXIT and its confidence bands relies on Theorem 2. However, we introduce a finite sample correction, justified by Theorem 1, which has no asymptotic impact.

In this paper, we always have $T = 1$ day (both on simulations and on data) and use evenly sampled returns to compute EXIT, so that $H'_{3/2} = 1$ and $H_2(T) = T$. Denote these returns by

$r_1, \ldots, r_n$, so that n is the number of returns in one day, and $\Delta = 1/n$. Define the *error function* as

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt.$$

We first need estimates of the spot volatilities. In the spirit of Fan and Wang (2008), we use the kernel estimator

$$\widehat{\sigma}_j^2 = \frac{\pi}{2} \frac{\sum_{i=1}^{n-1} K\left(\frac{i-j}{h_n}\right) |r_i||r_{i+1}| I_{\{r_i^2 \le \theta_i\}} I_{\{r_{i+1}^2 \le \theta_{i+1}\}}}{\Delta \sum_{i=1}^{n-1} K\left(\frac{i-j}{h_n}\right) I_{\{r_i^2 \le \theta_i\}} I_{\{r_{i+1}^2 \le \theta_{i+1}\}}}, \quad j = 1, \ldots, n \tag{B.1}$$

where the *threshold* $\theta_i,\ i = 1, \ldots, n$ is obtained as in Corsi et al. (2010) with $c_\theta = 5$. We set $h_n = 25$ and the function $K(\cdot)$ is a double-exponential kernel

$$K(x) = \frac{1}{2} e^{-|x|}.$$

As in Corsi et al. (2010), the bipower variation term $|r_i||r_{i+1}|$ combined with the threshold provides a jump-robust volatility estimator with satisfactory finite sample properties.

To estimate the variance of microstructure noise $\sigma_\varepsilon^2 = 2\sigma_\eta^2$, we use all available transaction prices. Denote by $p_1, \ldots, p_N$ the logarithmic prices observed in one day, so that N is the total number of transactions (typically $N >> n$). The microstructure noise variance estimator is (Bandi and Russell, 2006; Zhang et al., 2005)

$$\widehat{\sigma}_\varepsilon^2 = \frac{1}{N-1} \sum_{i=1}^{N-1} (p_{i+1} - p_i)^2 - \frac{\sum_{i=1}^n r_i^2}{N-1},$$

where the second term is a small-sample correction.

Now, write

$$\mathcal{P}_i = \operatorname{erf}\left(\frac{\xi_n}{\sqrt{2\left(\Delta\widehat{\sigma}_i^2 + \widehat{\sigma}_\varepsilon^2\right)}}\right),$$

where $\Delta\widehat{\sigma}_i^2$ is an asymptotically-vanishing finite-sample correction justified by Theorem 1. EXIT is then computed as

$$\text{EXIT} = \frac{1}{n} \sum_{i=1}^n \left(1_{\{|r_i| \le \xi_n\}} - \mathcal{P}_i\right), \tag{B.2}$$

while its variance V_{EXIT} is computed as

$$V_{\text{EXIT}} = \frac{1}{n^2} \sum_{i=1}^n \left(\mathcal{P}_i - \mathcal{P}_i^2\right). \tag{B.3}$$

Notice that in these expressions we use the erf function instead of its asymptotic equivalent in Theorem 1 and Theorem 2. The logic of (B.2) and (B.3) is immediate once it is recognized that the indicators amount to approximate Bernoulli random variables. The use of the term $\mathcal{P}_i^2$ is asymptotically irrelevant but empirically important in a finite sample.