Understanding and Exploiting
Momentum in Stock Returns

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First version: January 25, 2005. This version: September 2005

JEL-Classification: G11, G12, G13, G23.
Keywords: Momentum, stock returns, portfolio strategies.

Abstract

We study the asset-allocation consequences of momentum in a complete-markets setting starting from a novel continuous-time model in which the state variable is a weighted average of current and past returns on stocks. We isolate and discuss three clear effects of momentum on the optimal demand for stocks: the speculative effect, the conditional-hedge effect, and the unconditional-hedge effect.

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1 Introduction

The existence of a predictable component in stock returns is well documented and the ensuing autocorrelation of holding-period returns does impact optimal portfolio choice by originating strategic asset allocation. Positive (negative) autocorrelation is a clear-cut measure of momentum (mean reversion). Return autocorrelation has motivated well-known active portfolio management strategies, like either "buy winners / sell losers" to profit from momentum, or contrarian strategies to profit from mean reversion. These strategies are quite popular among practitioners and important in the academic assessment of the efficient markets hypothesis. However, although the asset-allocation literature has explored in depth the mean-reversion case, momentum has not received pairwise attention. The purpose of this paper is to study the continuous-time asset-allocation problem of an investor who, given a finite time horizon, tries to exploit momentum. We focus on the characterization of her optimal demand for the risky stock. Since momentum implies a stochastic opportunity set for the investor (changes in momentum imply changes in the instantaneous conditional expected returns on the stock), we are not only interested in unveiling the active component of her speculative demand, but also in providing a thorough description of her hedging demand. ‘Momentum-watching’ investors are typically concerned with the latest levels of stock returns relative to some target or long-run level. Therefore, we introduce a model of stock price dynamics in which the state variable is observable, related to current performance as well as past performance of returns (it is a weighted average of up-to-date returns), and conducive to positive autocorrelation of holding-period returns. This is a novelty in the strategic-asset-allocation literature (so termed by Brennan,
Lagnado, and Schwartz 1997), which has never explicitly included weighted averages of past and current returns in the conditioning-information set. Our model is particularly well suited to study momentum in a complete markets setting, which enables closed-form solutions and greatly facilitates the treatment of investor’s intermediate consumption. Chiefly, given momentum-like departures from the efficient market hypothesis, our model is a neat extension of strategic asset allocation to the study of how momentum impacts the speculative and hedging demands for the stock. We characterize a (weakly) efficient market as one in which the stock price is geometric Brownian motion (GBM).

In our closed-form analysis we isolate and discuss three clear effects of momentum on strategic asset allocation: the speculative effect, the conditional-hedge effect, and the unconditional-hedge effect. Regarding the speculative demand, momentum induces a myopic but active strategy of buying winners / selling losers, in which the momentum parameter of the stock price dynamics neatly appears. The sign of the conditional hedging demand depends on whether the latest performance of returns has been either above (negative demand) or below (positive demand) the long-run expected return. By boosting the volatility of distant-future stock prices, momentum makes the unconditional hedging demand always negative.

In continuous-time partial equilibrium, Kim and Omberg (1996) and Wachter (2002) constitute the two reference closed-form analyses of optimal finite-horizon portfolio choice in the presence of drift-based predictability (see also Wachter and Sangvinatsos (2005) and the references therein). They consider drift-based predictability by taking the market price of risk as the state variable. Wachter (2002) considers a complete-markets special case of the Kim-Omberg (1996) economy to enable the treatment of intermediate
consumption à la Cox and Huang (1989). Although there are clear similarities between our model and their reference models (they are discussed in Sections 7 and 8) we claim that our parameterization has important advantages. Chiefly, taking observable (current and past) returns as the state variable makes sure that momentum is introduced in a natural way ('past winners / loosers remain so') so that neat intuition accompanies the closed-form results we get. On these grounds, we escape a nontrivial pitfall of the existing theoretical literature on strategic asset allocation. As Campbell and Viceira (2002) have pointed out (p. 127): “While the theoretical literature has made considerable progress in recent years, the cases with known exact analytical solutions are still relatively few, and the solutions often have complicated forms that are hard to interpret...”. Following Wachter’s (2002) solution techniques, our complete-markets model can easily handle the case of intermediate consumption.

Our work is organized as follows. Sections 2, 3, and 4 review predictability of stock returns, its general impact on asset allocation, and its specific facet of momentum. Section 5 discusses our data-generating process and introduces the momentum state variable. Section 6 studies strategic asset allocation in the presence of momentum and with utility from terminal wealth only. Section 7 relates our model to the Kim and Omberg’ (1996) setting. Section 8 treats the intermediate-consumption case. After the concluding section, an Appendix gathers the technical details.

2 Predictability of stock returns

Until the mid 1980’s there was widespread consensus among financial economists that stock returns were unpredictable. Unpredictability was seen as a di-
rect consequence of efficient markets. However, this consensus started to be eroded by the works of Fama and French (1988) and Poterba and Summers (1988), who found statistical evidence that past returns were helpful to predict future returns. Actually, they showed that equity returns tend to mean-revert at long horizons.

Although the evidence provided by Fama and French (1988) and Poterba and Summers (1988) was rather weak, their research suggested a more general question: is it possible to find other variables, beyond past returns, that are useful to predict future returns? Efforts to answer this question gave rise to an explosion of research and to an entire new branch of financial literature.

A clear consequence of this new research is that, once additional variables are included into the analysis, the evidence of predictability looks much stronger than what Fama and French (1988) and Poterba and Summers (1988) have suggested. A surprisingly wide array of predictors has been found useful to forecast future returns. To cite just the most influential, they go from dividend-price ratios and earning-price ratios (Campbell and Shiller 1988, Fama and French 1988), changes in short term interest rates (Fama and Schwert 1977, Campbell 1991), yield spreads (Keim and Stambaugh 1986, Campbell 1987), to the consumption-wealth ratio (Lettau and Ludvingson 2001).

Nowadays the consensus seems to have changed to a wider acceptance of the idea that returns are actually predictable. Predictability has become a part of mainstream finance to such an extent that an authoritative textbook like Cochrane’s (2001) devotes a chapter to it, calling it a “new fact in finance”. Schwert (2003) also provides an extensive account of predictability.

Given that predictability is in the data, financial economists have inves-
tigated how it affects the multiperiod asset-allocation decisions of a rational long-lived investor.

3 Predictability and Asset Allocation

In continuous-time finance, the benchmark of unpredictability of stock returns is represented by the GBM model. The GBM model of stock’s instantaneous returns is:

\[
\frac{dS_t}{S_t} = \mu dt + \sigma dZ_t,
\]

(1)

where \(\mu\) is the (constant) expected return, \(\sigma\) is the instantaneous volatility, and \(Z_t\) is a Wiener process. In equation (1) there are no variables dated at time \(t\) that can be used to forecast returns between \(t\) and \(t + dt\). In this sense we say that returns are unpredictable.

The GBM model of returns has dramatic consequences for asset allocation. Suppose that we have an investor who derives expected utility from her terminal wealth. Her value function is:

\[
J(W_t, x, \tau) \equiv \sup_{\pi} E_t \left[ \frac{W_t^{1-\gamma}}{1-\gamma} \right],
\]

(2)

where \(W_t\) is her wealth at time \(t\) and \(\gamma\) is her Constant Relative Risk Aversion (CRRA) coefficient. Suppose that the investor knows that the correct dynamics of \(S_t\) is described by equation (1). The investor can trade only on the stock (which can be understood in this context as the market portfolio) and a riskless bond. Therefore, her wealth evolves according to:

\[
\frac{dW_t}{W_t} = r dt + \pi_t \left( \frac{dS_t}{S_t} - r dt \right)
\]

(3)
where \( \pi_t \) is the fraction of her wealth allocated to the stock at time \( t \) and \( r \) is the riskless interest rate (assumed constant). This is a standard problem in which the investor chooses \( \pi \) to achieve (2) subject to (3). The ensuing optimal policy is:

\[
\pi_t^* = \frac{1}{\gamma} \frac{\mu - r}{\sigma^2}. 
\] (4)

In equation (4) \( \pi^* \) is constant and independent of the investor’s horizon. The model predicts that the investor will allocate a constant fraction of her wealth to the stock. Moreover, this constant fraction depends on the Sharpe ratio of the stock and on the investor’s CRRA coefficient, but not on the investor’s horizon. This has the odd implication that two investors with the same attitude towards risk should allocate the same proportion of their wealth to the stock, even if the first investor has a horizon of one week and the second investor has a horizon of thirty years. Such a prediction casts doubts about the usefulness of the model as a means to provide advise to real investors.

In the GBM model, expected return and instantaneous volatility are constant. The riskless rate is also assumed to be constant. The whole situation is summarized by saying that the investor faces a constant investment opportunity set. Merton (1971) was the first to explore the consequences on asset allocation of relaxing the assumption of a constant investment opportunity set. He showed that these consequences are far-reaching. In particular, asset demands are no longer constant as a fraction of wealth and horizon effects may enter the picture.

Because predictability is a form of nonconstant investment opportunity set (expected returns are time-varying), we will illustrate Merton’s seminal contribution in the context of predictability. We keep the focus on pre-
dictability by retaining the assumption that both the instantaneous volatility and the riskless rate are constant.

Suppose that the stock price is driven by the following dynamics:

\[ \frac{dS_t}{S_t} = \mu(x_t) \, dt + \sigma dZ_t, \quad (5) \]

where \( x_t \) is a predictor that affects the conditional expected return \( \mu(x_t) \). The predictor \( x_t \) evolves according to:

\[ dx_t = \alpha(x_t) \, dt + \omega(x_t) dB_t, \quad (6) \]

with

\[ E_t(dZ_t dB_t) = \rho dt \quad (7) \]

being the instantaneous correlation of the Wiener processes \( B_t \) and \( Z_t \). The investor’s problem is still one of achieving (2), but now there exists a variable \( x_t \) that is instrumental to predict future returns. Because \( x_t \) determines the state of the investment opportunity set and stock returns are correlated to changes in \( x_t \), the investor is interested in hedging against unfavorable changes in the investment opportunity set. So, the optimal demand for the stock is:

\[ \pi^* = \frac{1}{\gamma} \left( \frac{\mu(x_t) - r}{\sigma^2} + \frac{J_{Wx} \rho \omega(x_t)}{-W J_{WW} \sigma} \right) \quad (8) \]

There are two new features in equation (8) relative to the GBM case. First, the speculative demand depends now on the predictor observed at time \( t \). This introduces the realistic possibility of market-timing strategies.
into the model. Market-timing strategies act through the conditional Sharpe ratio, which induces the investor to take a speculative position in the stock. Second, there is a new component, the intertemporal hedging demand, which is used to hedge against $x$-driven stochastic changes in the investment opportunity set. A good (bad) opportunity set is welfare-increasing (welfare-decreasing) because it reduces (raises) the marginal utility of wealth, $J_W$, by improving (worsening) the conditional Sharpe ratio, that is, the conditional ‘productivity’ of wealth. It is to be noticed that, as long as the marginal utility of wealth is affected by $x_t$ ($J_{Wx} \neq 0$), the intertemporal hedging demand would be zero only if $x_t$ were deterministic ($\omega = 0$), or if $x_t$ were uncorrelated to stock returns ($\rho = 0$).

Predictability has been studied in the asset allocation literature mainly in the form of time-varying expected returns (see, for example, Barberis 2001 and Campbell and Viceira 1999) and time-varying return volatilities (see Chacko and Viceira 2005). The modern asset allocation literature has tried to find a rationale to the known practitioner’s advice that the allocation to stocks should increase with the investor’s horizon, and, for this reason, it has concentrated on the study of mean-reverting returns.

Momentum represents a clear form of predictability of stock returns. Although momentum is a strong stylized fact of stock returns, it has been mostly disregarded by the strategic-asset-allocation literature—see Balvers (1997) for a discrete-time study of autocorrelated returns and optimal intertemporal portfolio choice. The next section is dedicated to reviewing how the phenomenon of momentum has been defined, measured, and understood in the existing literature.
4 Momentum

Campbell (2004) defines momentum to be the inclination of stock prices to keep on moving in the same direction for several months after an initial shock. Momentum gives rise to positive autocorrelation of certain holding-period returns. Price momentum occurs when the initial shock is a change in the price itself. Price momentum was found in aggregate US stock prices in the late 1980’s (Lo and MacKinlay 1988; Conrad and Kaul 1988; and Poterba and Summers 1988), in individual US stock prices in the early 1990’s (Jegadeesh and Titman, 1993), and in international markets later in the 1990’s (Rouwenhorst 1998, 1999). Different initial shocks yield other sorts of momentum. Post-earnings-announcement drift is momentum following a surprise earnings announcement (Ball and Brown 1968, Bernard and Thomas 1989, 1990), while earnings momentum is momentum following a revision in analysts’ earnings forecasts (Chan, Jegadeesh, and Lakonishok 1996).

Considerable evidence exists that momentum investment strategies produce excess returns. The work of Jegadeesh and Titman (1993), Chan, Jegadeesh, and Lakonishok (1996), Rouwenhorst (1998), Chan, Hameed and Tong (2000), Grundy and Martin (2001), Jegadeesh and Titman (2001), Lewellen (2002), Patro and Wu (2004), and others reveals that a momentum strategy of sorting (portfolios of) stocks by previous returns and holding those with the best prior performance and shorting those with the worst prior performance generates positive excess returns. Momentum strategies typically work for a sorting period ranging from 1 month (or more commonly 3 months) to 12 months and a similar 1 (or 3) to 12 months holding period. Balvers and Wu (2004) shows that combination momentum-contrarian
strategies, used to select from among several developed equity markets at a monthly frequency, outperform both pure momentum and pure contrarian strategies. A *caveat* is that profitable momentum strategies are hard to be implemented in extremely illiquid stocks (Korajczyk and Sadka 2004).

Explaining momentum within a classical asset pricing model is difficult. Such a model would explain momentum if stocks that have risen recently, or have had positive earnings surprises were to command high average returns by exhibiting higher risk. Grundy and Martin (2001), Griffin, Ji, and Martin (2003) find that this is hardly the case. A well known stylized fact like the leverage effect (the equity of a leveraged company becomes safer when it appreciates) is at odds with classical risk-based explanations of momentum.

Momentum is accommodated more easily within a behavioral asset pricing model, where momentum is explained as the result of the interaction of imperfectly rational investors with rational arbitrageurs. Behavioral explanations of momentum are either stories of underreaction to relevant news for the future cash flows of a stock or stories of overreaction to fickle news.

Investors with limited ability to access and process information and overconfident investors who cherish their original views even in the face of relevant new information are the prime cause of underreaction (Daniel, Hirshleifer, and Subrahmanyam 1998). Rational arbitrageurs do respond to fundamental news, but they trade gradually to unwind their positions profitably as, over time, the price adjusts fully to the news. Underreaction is consistent with the strong evidence for momentum in response to fundamental shocks such as earnings announcements or analysts’ forecast revisions.

Overreaction is irrational exuberance in the face of indecisive or qualitative information (Daniel and Titman 2004). Irrationality of this sort pushes prices away from the level that would be justified by fundamentals and can
be exploited by rational arbitrageurs. Momentum ensues if irrational investors respond gradually to qualitative information, if they imitate each others’ trades, or if they have a penchant for buying stocks that have performed well recently. ‘Herding’ is the term used to classify these behavior patterns. The herding hypothesis is consistent with evidence on flows into mutual funds (Sirri and Tufano 1998). This evidence suggests that individual investors are attracted to funds, fund categories, and fund families that have performed well recently.

Irrational enthusiasm can trigger a stock market bubble if rational arbitrageurs find that riding positive short-term momentum is more profitable than selling off on the basis of poor long-term value. Brunnermeier and Nagel (2004) use this argument to show that hedge funds rode the technology bubble through the late 1990’s even after technology stocks became recklessly overpriced.

The behavioral model of momentum implies that momentum should be stronger (i) when fundamental news is less obvious and harder to analyze (Zhang 2004 finds stronger momentum in stocks that are hard to value and Grinblatt and Moskowitz 2004 in cases when news comes out slowly over several months); (ii) when other behavioral forces push in the same direction (Grinblatt and Moskowitz 2004 find that tax-loss selling strengthens momentum in December and weakens it in January); and (iii) when rational investors face high transactions costs in their arbitrage trading (Johnson and Schwartz 2000 find that momentum weakened in liquid markets such as the US and the UK).

In the next section we model momentum in continuous time as a weighted average of past and current returns on the stock that gravitates around the stock’s unconditional expected return.
The data generating process

We make the following specification of stock’s instantaneous returns:

\[
\frac{dS_t}{S_t} = y_t dt + dy_t, \quad (9)
\]

\[
dy_t = -(1 - \phi) (y_t - \mu) dt + \sigma dz_t, \quad 0 < \phi < 1, \quad (10)
\]

\[
\mu > 0, \quad \sigma > 0,
\]

\[
E \left[ \frac{dS_t}{S_t} \right] = \mu dt \quad \text{(unconditional expected return)},
\]

\[
Var \left[ \frac{dS_t}{S_t} \right] = \sigma dt \quad \text{(return’s unconditional variance)},
\]

where \(z_t\) is a Wiener process. The long-run expected return \(\mu\) is above the constant interest rate \(r\), \(\mu > r > 0\) so that the unconditional Sharpe ratio \(\frac{\mu - r}{\sigma}\) is positive. By construction, the momentum state variable is a weighted average of past stock returns:

\[
y_t = \int_0^t \exp(u - t) \frac{dS_u}{S_u}, \quad dy_t = \frac{dS_t}{S_t} - y_t dt.
\]
The pivotal role of the predictability parameter $\phi$ can be seen in this re-expression of the stock’s instantaneous returns:

$$\frac{dS_t}{S_t} = (\mu + \phi (y_t - \mu)) \, dt + \sigma dz_t. \quad (11)$$

If past performance has been above its long-run average ($y_t - \mu > 0$), stock returns are expected to be above their unconditional average ($E_t \left( \frac{dS_t}{S_t} \right) > \mu$).

GBM results from the case $\phi \downarrow 0$, so that predictability drops out.

A further enhancement of our intuition of the model comes from analyzing the unconditional moments of discrete holding period returns. Define $\Delta S_t = \log (S_{t+\tau}) - \log (S_t)$:

$$E(\Delta S_t) = \left( \mu - \frac{1}{2} \sigma^2 \right) \tau,$$

$$Var(\Delta S_t) = \frac{\sigma^2}{(1-\phi)^2} \left[ \tau + \frac{\phi}{1-\phi} \left( 1 - e^{-(1-\phi)\tau} \right) (\phi - 2) \right],$$

$$Cov(\Delta S_t, \Delta S_{t-\tau}) = \frac{\sigma^2}{(1-\phi)^2} \left[ \frac{\phi}{1-\phi} \left( 1 - e^{-(1-\phi)\tau} \right)^2 \left( 1 - \frac{\phi}{2} \right) \right],$$

$$\rho_1(\Delta S_t, \Delta S_{t-\tau}) \equiv \frac{Cov(\Delta S_t, \Delta S_{t-\tau})}{Var(\Delta S_t)} = \frac{\frac{\phi}{1-\phi} \left( 1 - e^{-(1-\phi)\tau} \right)^2 \left( 1 - \frac{\phi}{2} \right)}{\frac{\phi}{1-\phi} \left( 1 - e^{-(1-\phi)\tau} \right) (\phi - 2) \tau + \frac{\phi}{1-\phi} \left( 1 - e^{-(1-\phi)\tau} \right) (\phi - 2)}.$$

The variance of the holding period return grows without bound for all values of $\phi$ as the holding period goes to infinity, reflecting that the stock
price is a nonstationary process. However, the variance’s growth rate does depend on $\phi$. The variance grows linearly with the horizon when $\phi \downarrow 0$ (the stock return is a random walk with drift) and faster than linearly when $\phi > 0$ (momentum). When $\phi > 0$, the variance of the holding period return is higher than the variance of the random walk case, for all finite $\tau$. This latter property will have implications for long term investors. When $\phi > 0$, long term investors will have less stocks in their portfolios, relative to the random walk case. This is because they will see stocks as increasing the risk of their portfolios when there is momentum.

The covariance and first autocorrelation of the holding period return have the same sign as $\phi$. When $\phi > 0$, momentum generates positive autocorrelation. The closer is $\phi$ to 1, the more pronounced is momentum and the more intense is positive autocorrelation. It deserves to be re-stated that, in short holding period returns (e.g., daily, weekly, and monthly stock index returns), several studies report that there exist strong positive autocorrelations (see Lo and MacKinlay, 1988; Conrad and Kaul, 1988; and Poterba and Summers, 1988). When $\phi \downarrow 0$, autocorrelation disappears as predictability fades away to bear the case of random walk with drift.

6 Strategic asset allocation

After the removal of its long-run average, the state variable becomes (we drop the time subscript $t$ unless its presence is necessary),

\[
x \equiv y - \mu.
\]
The problem is

\[ \text{the objective (2)} \]

\[ \text{s.t. (3), } \quad \frac{dS}{S} = (\mu + \phi x) \, dt + \sigma dz, \quad dx = -(1 - \phi) \, xdt + \sigma dz. \]  

Problem (12) involves solution steps that, although standard (see Kim and Omberg, 1996), shed much light on the structure of the optimal policy \( \pi^* \).

The Hamilton-Jacobi-Bellman (HJB) equation reads

\[ \sup_{\pi} E_t [dJ] = 0, \]

and the First Order Conditions give

\[ \pi^* = \frac{J_W}{-W J_{WW}} \frac{\mu - r + \phi x}{\sigma^2} + \frac{J_{Wx}}{-W J_{WW}}. \]  

(13)

so that the HJB equation evaluated at the optimal portfolio (13) becomes

\[ 0 = -J_r + J_W W_r + J_x (-(1 - \phi) x) \]

\[ -\frac{1}{2} J_{WW} (W \pi^* \sigma)^2 + \frac{1}{2} J_{xx} \sigma^2, \]  

(14)

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with boundary condition \( J(W, x, 0) = \frac{W^{1-\gamma}}{1-\gamma} \).

The following trial solution for \( J(W, x, \tau) \) implies that the horizon-end condition and the second-order condition for a maximum are satisfied:

\[
J(W, x, \tau) = I(x, \tau) \frac{W^{1-\gamma}}{1-\gamma},
\]

\[
I(x, \tau) \equiv \exp \left( A_1(\tau) \frac{x^2}{2} + A_2(\tau) x + A_3(\tau) \right),
\]

\[
A_1(0) = A_2(0) = A_3(0) = 0.
\]

Substitution of the trial solutions into the optimal portfolio gives

\[
\pi^* = \frac{1}{\gamma} \frac{\mu - r}{\sigma^2} + \frac{1}{\gamma} \frac{\phi x}{\sigma^2} + \frac{I_x}{I},
\]

where

\[
\frac{I_x}{I} = \frac{J_{Wx}}{-WJ_{WW}} = A_1(\tau) x + A_2(\tau).
\]

Therefore, we can decompose the overall optimal demand for stock into three portfolios:
The myopic unconditional portfolio is the well-known solution (4) of the GBM-investment problem with nonstochastic opportunity set. It can be interpreted in our context as a reference portfolio to which the investor will add short or long speculative positions on the stock depending on whether conditioning information implies negative or positive extra expected returns ($x < 0$ and $x > 0$). The myopic active portfolio does nest momentum strategies: buy winners ($x > 0$) and sell losers ($x < 0$) to profit from momentum ($\phi > 0$).

The hedging demand exploits the perfect positive correlation between stocks returns and the momentum innovation,

$$\frac{\text{cov} \left( \frac{dS}{S}, dx \right)}{\sqrt{\text{var} \left( \frac{dS}{S} \right) \text{var} (dx)}} = 1,$$

to create positive (negative) returns when $J_W$ is increased (decreased) by a negative (positive) shock on the investment opportunity set.
6.1 The closed-form solution

Substitution of the trial solution (15) and of the optimal portfolio (18) into the partial differential equation (14) gives

\[
0 = -\frac{\gamma}{1 - \gamma} \left( \frac{dA_1}{d\tau} \frac{x^2}{2} + \frac{dA_2}{d\tau} x + \frac{dA_3}{d\tau} \right) + r \\
+ \frac{\gamma}{1 - \gamma} (A_1 x + A_2) (- (1 - \phi) x) - \frac{1}{2} (-\gamma) \left( \frac{1}{\gamma} \frac{\mu - r + \phi x}{\sigma^2} + A_1 x + A_2 \right)^2 \sigma^2 \\
+ \frac{1}{2} \frac{\gamma}{1 - \gamma} (\gamma (A_1 x + A_2)^2 + A_1) \sigma^2.
\]

The last equation is a quadratic equation for \(x\). Since it must hold for any \(x\) and for any \(\tau\), its three coefficients must be zero, resulting in the system of first-order nonlinear ordinary differential equations:

\[
\frac{dA_1}{d\tau} = cA_1^2 + bA_1 + a, \quad (19) \\
\frac{dA_2}{d\tau} = cA_1 A_2 + b \frac{A_2}{2} A_1 + f A_1 + a f \frac{1}{\phi} \frac{\gamma}{1 - \gamma}, \quad (20) \\
\frac{dA_3}{d\tau} = \frac{c}{2} A_2^2 + \frac{c}{2} A_1 + f A_2 + f \frac{r}{\mu - r} + \frac{1}{2} a f^2 \left( \frac{1}{\phi} \frac{\gamma}{1 - \gamma} \right)^2, \quad (21)
\]

\[
a \equiv \frac{1 - \gamma}{\gamma^2} \left( \frac{\phi}{\sigma} \right)^2, \quad b \equiv 2 \left( - (1 - \phi) + \frac{1 - \gamma}{\gamma} \phi \right), \\
c \equiv \sigma^2, \quad f \equiv \frac{1 - \gamma}{\gamma} (\mu - r),
\]
with the initial condition (17).

When \( \gamma > 1 \), the quantity \( q \),

\[
q \equiv b^2 - 4ac,
\]
is always positive. Defining

\[
\eta \equiv q^{\frac{1}{2}},
\]
the solutions to (19) and (20) are given by (see Appendix)

\[
A_1(\tau) = \frac{2a (1 - \exp(-\eta \tau))}{2\eta - (b + \eta)(1 - \exp(-\eta \tau))},
\]

\[
A_2(\tau) = A_1(\tau) \left( \frac{1}{1 - \phi} + \frac{2}{\eta} \frac{(1 - \exp(-\eta \tau))^2}{(1 - \exp(-\eta \tau))} \right) \frac{\gamma}{1 - \gamma} \frac{1 - \phi}{\phi}.
\]

The solution to (21) does not enter the optimal policy (18) so that its study is not required to understand momentum-driven portfolio selection. When \( \gamma > 1 \), \( A_1(\tau) \) is always negative because \( a < 0 \) and \( 2\eta - (b + \eta) > 0 \). When \( \gamma > 1 \), \( A_2(\tau) \) is negative because \( \phi \in (0, 1) \) and \( f \frac{\gamma}{1 - \gamma} < 0 \). The clear-cut sign of \( A_1(\tau) \) and \( A_2(\tau) \) and the perfect positive correlation between stock returns and momentum innovations enable a neat interpretation of the hedging demand.
If the momentum variable is non-negative \((x \geq 0)\), the hedging demand will be always negative. Indeed, a long term investor desires negative (positive) returns in case of a positive (negative) shock in the momentum variable. This is because a positive (negative) shock in the momentum variable makes the investor better (worse) off by pushing down (up) the marginal utility of wealth, \(J_W\), via an improvement of (damage to) the investment opportunity set.

If the momentum variable is moderately negative \(\left(\frac{A_2(\tau)}{A_1(\tau)} < x \leq 0\right)\), the hedging demand will remain negative for the same argument above.

If the momentum variable is sufficiently negative \(x < \frac{A_2(\tau)}{A_1(\tau)}\), the hedging demand becomes positive. This is because extreme values of \(x\) (even if negative) do imply good investment opportunities so that a positive (negative) shock in the momentum variable makes the investor worse (better) off by damaging (improving) the opportunity set.

**Figure 1**

![Graph](image-url)

- Myopic active p. \((x = -20\%\)
- Hedging portfolio \((x = -20\%\)
For $\gamma = 5$, Figures 1, 2, and 3 plot the myopic active portfolio and the hedging portfolio against the investor's horizon $\tau$. The other parameters are: riskfree rate $r = 3\%$, unconditional risk premium $\mu - r = 7\%$, volatility $\sigma = 30\%$, momentum parameter $\phi = 0.255$ (which implies a first autocorrelation of 1-year holding-period returns of $\rho_1 = 12\%$). The myopic unconditional
portfolio is 15.56%.

In Figure 1, \( x = -20\% \), which implies that returns have been on average 20% below their long run level of 10%. Because there is momentum, the investor expects further negative returns (“losers remain so”). This expectation induces a negative myopic active portfolio (a short position) equivalent to about 11% of wealth, and an overall negative hedging portfolio equivalent, for a very long-lived investor, to about 1.75% of wealth. In this case, the conditional component \( A_1(\tau) x \) and the unconditional component \( A_2(\tau) \) of the hedging demand work in opposite directions. The conditional component is positive, curbing the long position generated by the myopic active position, while the unconditional component is negative, curbing the short position generated by the long run myopic position (the unconditional Sharpe ratio is positive in this example). In Figure 2 returns have been on average at their long-run level, and so the myopic active portfolio is zero, while the hedging portfolio is negative reflecting only the unconditional component of the hedging demand. Figure 3 shows the case in which returns have been on average 20% above their long-run level. In this case, the myopic active portfolio is positive (about 11% of wealth) as the investor expects “winners to remain so”. The hedging portfolio is strongly negative (about 4% of wealth of a very long-lived investor), because in this case the unconditional component and the long run component of the hedging demand work in the same direction.

7 Recasting Kim and Omberg (1996)

Our setting recasts the Kim and Omberg’ (1996) model with a number of important restrictions. Their importance lies in that they are conducive to
a state variable that is a clear signal for momentum (\( y \) is a weighted average of past and current returns on the stock) and to a neat interpretation of the resulting optimal policy (as discussed in the previous section). In Kim and Omberg (1996), the state variable is the market price of risk \( X \), with dynamics

\[
dX = -\lambda_X (X - \overline{X}) \, dt + \sigma_X dz_X. \tag{22}
\]

The market price of risk enters the stock price dynamics in this fashion:

\[
\frac{dS}{S} = (r + X \sigma) \, dt + \sigma dz. \tag{23}
\]

Kim and Omberg (1996) fix the annualized instantaneous covariance \( \frac{1}{dt} \text{Cov}(dz_X, dz) \) to be the parameter \( \rho_{mX} \). Our assumption of market completeness implies

\[
\rho_{mX} = 1.
\]

The comparison of the two stock price dynamics (11) and (23) implies these three restrictions on the dynamics of the market price of risk,

\[
\lambda_X = 1 - \phi, \quad \overline{X} = \frac{\mu - r}{\sigma}, \quad \sigma_X = \phi
\]
together with the following relationship

\[ X = \frac{\mu - r}{\sigma} + \frac{\phi}{\sigma} (y - \mu). \]

8 Intermediate consumption

Our complete-markets model allows the use of Cox-Huang’ (1989) martingale approach to solve the case with intermediate consumption, \( c \), as in Wachter (2002). The problem is

\[
J(W, x, \tau) \equiv \sup_{\pi} E_t \left[ \int_0^\tau \exp(-\rho s) \frac{c_t^{1-\gamma}}{1-\gamma} ds \right],
\]

s.t.

\[
dW = W \left( r dt + \pi \left( \frac{dS}{S} - r dt \right) \right) - cd t,
\]

\[
\frac{dS}{S} = (\mu + \phi x) dt + \sigma dz,
\]

\[
dx = -(1 - \phi) x dt + \sigma dz.
\]

This is the problem of an investor who still does market timing, but that at the same time consumes out of her wealth. The martingale approach
to this problem states that, since the investor’s wealth $W = W(Z,x,t)$ is a tradable asset, it must obey a no-arbitrage condition. In combination with a trial solution for $W(Z,x,t)$, such a condition generates a system of ordinary differential equations akin to that discussed in Section 6. For brevity we do not provide details of the solution method. These details can be found in Wachter (2002). The resulting optimal demand for the stock is:

$$\pi^* = \frac{1}{\gamma} \left( \frac{\mu - r}{\sigma^2} + \frac{\phi x}{\sigma^2} + \int_0^\tau \frac{I(x,s)}{I(x,v)} [A_1(s) x + A_2(s)] ds \right),$$

where $I(x,\tau)$ is as in definition (16). The functions $A_i(\tau)$, $i = 1, 2, 3$ are the solutions of the terminal-wealth-case equations (19), (20), and (21). There is also no change in the speculative portfolios relative to the terminal-wealth case. In particular, active momentum strategies remain the same. The main difference between the two models is in the hedging portfolios, which are now a weighted average of the terminal-wealth-case hedging portfolios. Since $I(x,\tau)$ represents the value, scaled by today’s consumption, of consumption in $\tau$ periods, the weights are the present values of future consumption at any time $\tau$ relative to the present value of the entire flow of consumption.

9 Conclusions

Active portfolio-management strategies suggest that, in the presence of autocorrelation of holding-period returns on the stock, the observation of current returns may induce changes in the composition of optimally chosen portfo-
lios. In this paper we present a model of stock price dynamics in which the state variable is a weighted average of current and past returns. Our choice of model allows us to study in closed form the asset-allocation consequences of momentum in a complete-markets setting, so to extend the literature on intertemporal portfolio selection with predictability to the analysis of complex momentum strategies adopted by rational investors who try to exploit existing departures from the efficient market hypothesis. We isolated and discussed three clear effects: The speculative effect, the conditional-hedge effect, and the unconditional-hedge effect.

A first natural extension of our model is to relax the assumption that the investor observes the long-run expected return, and to explore instead a situation in which she must estimate it using current information. Estimation risk may enhance the role of the hedging demand for the stock. A second natural extension is to relax the assumption of absence of model uncertainty. Model uncertainty and aversion to it are known to trigger an additional hedging-demand component that is not linked to \( J_{W_x} \) but to \( J_x \) instead (see for example Sbuelz and Trojani 2004). This is likely to generate more interaction between the hedging demand and momentum-like predictability. Finally, we have studied the momentum case in isolation, disregarding the existence of mean reversion in long-holding-period returns on the stock. The interplay of both stylized facts in the asset-allocation decision of a rational investor is an issue that deserves investigation. The very recent work of Koijen, Rodriguez, and Sbuelz (2005) is the first to explicitly tackle the issue. This is the direction of our current research.
APPENDIX

Solution for $A_1$

The equation for $A_1$ is known as a Riccati equation and can be rewritten in integral form as

$$
\int_0^\tau \frac{dA_1}{cA_1^2 + bA_1 + a} = \tau.
$$

We start from the differential form (19):

$$
\frac{dA_1}{d\tau} = cA_1^2 + bA_1 + a.
$$

Let’s call $Y = \frac{1}{A_1}$. Note that $Y$ also satisfies the following Riccati equation:

$$
\frac{dY}{d\tau} = AY^2 + BY + C,
$$

where $C = -c$, $B = -b$, and $A = -a$. It is a known result that if $u(\tau)$ is a solution the Riccati equation (24), then the general solution of (24) is

$$
Y(\tau) = u(\tau) + \frac{1}{z(\tau)},
$$

where $z(\tau)$ satisfies a linear ordinary differential equation. Notice that a constant solution of (24) is

$$
u = \frac{-B + \sqrt{B^2 - 4AC}}{2A}.
$$
Therefore, the general solution of (24) is \( Y(\tau) = u + \frac{1}{z(\tau)} \), with:

\[
\frac{dz}{d\tau} = -z\sqrt{q} - A,
\]

where \( q = B^2 - 4AC = b^2 - 4ac \).

This last equation must be solved with initial condition \( z(0) = 0 \) (this is implied by the condition that \( A_1(0) = 0 \)):

\[
z(\tau) = -\frac{A}{\sqrt{q}} \left( 1 - e^{-\sqrt{q}\tau} \right)
\]

The general solution of \( Y \) is

\[
Y(\tau) = \frac{-B + \sqrt{q}}{2A} - \frac{\sqrt{q}}{A(1 - e^{-\sqrt{q}\tau})}.
\] (25)

Recall that \( B = -b \) and \( A = -a \), and that \( A_1 = \frac{1}{\sqrt{q}} \). Plugging this information in the general solution (25) gives (after some additional algebraic manipulation):

\[
A_1(\tau) = \frac{2a \left( 1 - e^{-\sqrt{q}\tau} \right)}{2\sqrt{q} - (b + \sqrt{q}) \left( 1 - e^{-\sqrt{q}\tau} \right)}
= \frac{2a (1 - \exp(-\eta\tau))}{2\eta - (b + \eta) \left( 1 - \exp(-\eta\tau) \right)},
\]

which is our stated expression for \( A_1(\tau) \).

Solution for \( A_2 \)
Given the restriction $\gamma > 1$, the closed-form solution for $A_2 (\tau)$ comes from knowing the closed forms for $A_1 (\tau)$ and the auxiliary function $A^*_2 (\tau)$. The auxiliary function $A^*_2 (\tau)$ solves the following Cauchy problem,

$$\frac{dA^*_2}{d\tau} = cA_1 A^*_2 + \frac{b}{2} A^*_2 + fA_1,$$

$$A^*_2 (0) = 0,$$

with analytical solution (see Kim and Omberg 1996, p. 147 and p. 158)

$$A^*_2 (\tau) = \frac{2}{\eta} \cdot A_1 (\tau) \cdot f \cdot \frac{(1 - \exp (-\frac{\eta}{2} \tau))^2}{(1 - \exp (-\eta \tau))}.$$

We formulate a trial solution for $A_2 (\tau)$ as a linear combination of $A_1 (\tau)$ and $A^*_2 (\tau)$:

$$(A_1 (\tau) w + A^*_2 (\tau)) u,$$

By construction, such a trial solution substitution satisfies the initial condition $A_2 (0) = 0$. Substitution of the trial solution into $A_2 (\tau)$’s differential equation (20) implies that the suitable values for the coefficients $w$ and $u$
\[
\begin{align*}
    w &= f \cdot \frac{1}{1 - \phi}, \\
    u &= \gamma \frac{1 - \phi}{1 - \gamma \phi},
\end{align*}
\]

which renders our stated expression for \( A_2(\tau) \).
REFERENCES


Campbell, J. Y. and Viceira, L. M. (1999). Consumption and portfolio decisions when expected returns are time varying. Quarterly Journal of


Daniel, Kent and Sheridan Titman, 2004, Market reactions to tangible and intangible information, unpublished paper, Northwestern University and University of Texas.


