Hedging double barriers with singles

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Abstract

Double barrier options can be statically hedged by a portfolio of single barrier knockin options. The main part of the hedge automatically turns into the desired contract along the double barrier corridor extrema.

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Keywords: Double barrier options, single barrier options, static hedging.

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1. Introduction

Barrier derivatives are the most liquid among the over-the-counter derivatives. European continuously-monitored barrier options are European options with an American feature. Option's existence depends on whether the underlying price breaches, before or at maturity, some prespecified levels, called barriers. Given one barrier, single knockin options come to life and single knockouts expire if the barrier is hit. Given two barriers, the double barrier corridor encompasses the initial underlying price. Double knockins come to life and double knockouts expire if either barrier is hit. A portfolio of a knockin and a knockout written on the same barriers and strike is equivalent to a vanilla option with the same strike. Thus, one can focus on knockins only.

Barrier options are very popular because they are cheaper than their vanilla counterparts. Via double barriers, investors enjoy even greater leverage potential. A double knockin may be used by a trader who foresees a lower/higher volatility than the market consensus one in both bullish and bearish scenarios.

The double barrier clause states: if either barrier is hit. This creates a double barrier interdependence and makes pricing and hedging difficult: A double knockin is not simply the sum of two single knockins written on the corridor extrema. I call that sum the Basic Portfolio. The Basic Portfolio is a super-replicating hedge: if the upper (lower) barrier is hit first, the single barrier contract written on the lower (upper) barrier contributes positive unwanted value. The hedger needs to add extra layers to get exact replication.

This work shows that, under the Black-Scholes assumptions, double barrier interdependence commands extra hedging layers all made of single knockins with the same maturity as the double barrier knockin.

The following numerical example shows the structure of those hedging layers. The current underlying price is $90. Consider a double knockin call with lower barrier $80 and upper barrier $100. Given a strike of $90, the double knockin call\(^1\) is priced $12.8079. The double knockin price is mainly made of the $100-in price ($12.7587) as the risk-adjusted logprice drift is positive and the probability of reaching the $100 level first is high. The following table shows barriers, strikes, portfolio amounts, portfolio amounts in $s of the single knockin positions that constitute

\(^1\)Other option parameters are: annualized riskfree rate equal to 5%, logprice annualized volatility equal to 30%, 1-year maturity, and payout rate equal to 0.
An example of Double Barrier Exact Hedge (DBEH)

<table>
<thead>
<tr>
<th>Knockin barrier in $s</th>
<th>Strike in $s</th>
<th>Amount</th>
<th>Amount in $s</th>
</tr>
</thead>
<tbody>
<tr>
<td>381.47</td>
<td>343.32</td>
<td>0.2434</td>
<td>0.000015</td>
</tr>
<tr>
<td>305.18</td>
<td>343.32</td>
<td>-0.2434</td>
<td>-0.000023</td>
</tr>
<tr>
<td>244.14</td>
<td>219.73</td>
<td>0.3898</td>
<td>0.009642</td>
</tr>
<tr>
<td>195.31</td>
<td>219.73</td>
<td>-0.3898</td>
<td>-0.011746</td>
</tr>
<tr>
<td>156.25</td>
<td>140.63</td>
<td>0.6243</td>
<td>0.835973</td>
</tr>
<tr>
<td>125.00</td>
<td>140.63</td>
<td>-0.6243</td>
<td>-0.887747</td>
</tr>
<tr>
<td>(upper barrier) 100.00</td>
<td>90.00</td>
<td>1.0000</td>
<td>12.758694</td>
</tr>
<tr>
<td>(initial spot price) 90.00</td>
<td>(original strike) 90</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(lower barrier) 80.00</td>
<td>90.00</td>
<td>1.0000</td>
<td>3.757592</td>
</tr>
<tr>
<td>64.00</td>
<td>57.60</td>
<td>-1.6017</td>
<td>-3.667750</td>
</tr>
<tr>
<td>51.20</td>
<td>57.60</td>
<td>1.6017</td>
<td>0.113625</td>
</tr>
<tr>
<td>40.96</td>
<td>36.86</td>
<td>-2.5655</td>
<td>-0.100540</td>
</tr>
<tr>
<td>32.77</td>
<td>36.86</td>
<td>2.5655</td>
<td>0.000559</td>
</tr>
<tr>
<td>26.21</td>
<td>23.59</td>
<td>-4.1093</td>
<td>-0.000413</td>
</tr>
<tr>
<td>20.97</td>
<td>23.59</td>
<td>4.1093</td>
<td>0.000000</td>
</tr>
</tbody>
</table>

Sum of the amounts in $s = Double knockin price

12.807870

The table illustrates how these single barrier options have barriers that take progressive distance from the original barrier corridor $80 / $100. Summing the portfolio amounts in $s of all the first 14 single knockins (from barrier level $21 to barrier level $381) gives the exact price of the double knockin. The Basic Portfolio, the sum of a $80-in and a $100-in only, is priced $16.5163. The full replicating portfolio is made of a countable infinity of single knockin positions and I call it Full Double Barrier Hedge (FDBH). The first few positions of the FDBH are sufficient to achieve good
replication of the double knockin.

The FDBH contributes along these lines. (1) It is static. (2) It exhibits an automatically-in feature along the barriers, because it has barrier-like nature as its target contract. (3) It takes account of the drift towards either barrier generated by a non-trivial cost of carrying the underlying asset. (4) It establishes an explicit link between single barrier pricing and double barrier pricing.

Portfolio amounts of the FDBH are static, that is, they do not depend on the underlying price and on the calendar time. Static hedging has the advantage of suffering less from transaction costs and pricing model misspecification as the hedger trades at most two times at the inception and, if necessary, at the first passage time of the underlying price through either barrier. The path-dependent options here examined often have high gammas and vegas, that is, their delta is highly time-varying and option prices are quite sensitive to volatility changes. In this case, static hedging is much likely to be easier and cheaper than dynamic hedging. The first analysis of static hedging of path-dependent options is due to Bowie and Carr (1994) and Derman, Ergener, and Kani (1994). Dupont (2001) employs a mean-square-hedging technique designed to minimize the size of the hedging error when perfect static replication of barrier options is not possible.

Static hedging of double barrier options by means of non-barrier options has been proposed by Carr, Ellis and Gupta (1998) (CEG), by Andersen, Andreasen, and Eliezer (2000), and by Forde (2004). Along the barriers, the hedger should fully unwind the hedge because the double barrier contract is automatically triggered. The main element of FDBH is the Basic Portfolio. Thus, if $100 is reached before $80 in due time, the $100-in leg automatically kicks in. With the FDBH, the hedger must only unwind its non-triggered legs.

If the underlying asset commands a positive (negative) cost of carry, then its risk-adjusted price exhibits a drift towards the upper barrier (lower barrier). Even in the presence of such a non-trivial risk-adjusted drift, the FDBH remains exact. I show that, with zero cost of carry, the FDBH specializes to the hedge proposed by CEG. This is because, if one breaks down the FDBH legs into subportfolios of non-barrier options, the two hedges correspond layer by layer. The FDBH needs a countable infinity of single knockins while the hedge proposed by Andersen, Andreasen, and Eliezer (2000) handles general price-dependent volatility but needs an along-all-strikes continuum of European options and an along-all-maturities continuum of calendar spreads.

Single barrier option prices are well known (see Merton (1973), Cox and Rubinstein (1985),
Benson and Daniel (1991), Hudson (1991), Reiner and Rubinstein (1991), Heynen and Kat (1994), Rich (1994), and Trippi (1994)). However, double barrier pricing is difficult because of the double barrier interdependence. The mathematics that unravels such an interdependence is awkward, so that existing closed-form prices (Douady (1999), Hui (1996), Hui, Lo, and Yuen (2000), Kunitomo and Ikeda (1992), Lin (1998), Pelsser (2000)) achieve elegance at the expenses of financial intuition. The FDBH states that the double barrier option price\(^2\) is a weighted sum of single barrier option prices with weights that do not depend on the underlying price and on the calendar time.

Geman and Yor (1996) and Jamshidian (1997) study the time-horizon Laplace transforms of double barrier option prices and suggest numerical inversion procedures to revert to prices. My analysis fully develops the financial-engineering potential of those works by carving out explicit pricing and static-hedging results by means of time-horizon Laplace transforms.

The rest of this article is organized as follows. Section 2 shows how the FDBH works. Section 3 concludes. The appendix gives technical details and proofs of the propositions.

### 2. The full double barrier hedge

I show that, under the Black-Scholes assumptions, the double barrier option price is a weighted sum of single barrier option prices. This novel pricing result casts new light on the financial nature of the contract by projecting the risk of double barrier instruments on to single barrier instruments.

Let \( C_{L,Knockin}^L(S_0, K, T) \) (\( C_{Knockin}^U(S_0, K, T) \)) denote the price of a single knockin call with barrier \( L \) (\( U \)). The three arguments of the price function are the initial price \( S_0 \) of the underlying asset, the strike price \( K \) (\( L \leq K \leq U \)), and the option maturity \( T \). The lower barrier \( L \) and upper barrier \( U \) straddle the initial underlying price \( S_0 \), \( L < S_0 < U \). The double knockin call, with price

\[
C_{Knockin}^{L,U}(S_0, K, T),
\]

is a call option which is initiated whenever either the upper barrier \( U \) or the lower barrier \( L \) is touched before or at option maturity. The instantaneous return rate of the riskfree asset is the constant \( r \) and the underlying asset offers a constant instantaneous payout rate \( d \). \( C(S_0, K, T) \) denotes the vanilla call price.

\(^2\)Perpetual double barrier option pricing has been recently used by Sbuelz(2004a) and Sbuelz (2004b) to study real options problems with payout-rate switching and with sunk-cost softening, respectively.
The FDBH unravels the pricing and hedging difficulty of double barrier options in a way that makes it easily comparable with the existing double barrier option literature, in particular with the double barrier option decomposition of Carr, Ellis and Gupta (1998).

**Proposition 1** Under the Black-Scholes assumptions, the double knock in call price has the following exact decomposition:

\[
C_{\text{knock in}}^{L,U}(S_0, K, T) = \left( 'FDBH' \right)
\]

\[
C_{\text{knock in}}^U(S_0, K, T) + C_{\text{knock in}}^L(S_0, K, T) + \left( 'Basic Portfolio' \right)
\]

\[
\sum_{n=1}^{\infty} \left( \frac{m_{BS}(0, \ln L, \ln U)}{m_{BS}(0, \ln U, \ln L)} \right)^n \times \left( \frac{U}{L} \right)^{+2n} \times C_{\text{knock in}}^L \left( S_0, K \left( \frac{U}{L} \right)^{-2n}, T \right)
\]

\[
\left( 'U-First-L-In Portfolio (UFLI), Part I' \right)
\]

\[
\sum_{n=1}^{\infty} \left( \frac{m_{BS}(0, \ln U, \ln L)}{m_{BS}(0, \ln L, \ln U)} \right)^n \times \left( \frac{U}{L} \right)^{-2n} \times C_{\text{knock in}}^U \left( S_0, K \left( \frac{U}{L} \right)^{+2n}, T \right)
\]

\[
\left( 'U-First-L-In Portfolio (UFLI), Part II' \right)
\]

\[
\sum_{n=1}^{\infty} \left( \frac{m_{BS}(0, \ln U, \ln L)}{m_{BS}(0, \ln L, \ln U)} \right)^n \times \left( \frac{U}{L} \right)^{-2n} \times C_{\text{knock in}}^U \left( S_0, K \left( \frac{U}{L} \right)^{-2n}, T \right)
\]

\[
\left( 'L-First-U-In Portfolio (LFUI), Part I' \right)
\]

\[
\sum_{n=1}^{\infty} \left( \frac{m_{BS}(0, \ln L, \ln U)}{m_{BS}(0, \ln U, \ln L)} \right)^n \times \left( \frac{U}{L} \right)^{+2n} \times C_{\text{knock in}}^U \left( S_0, K \left( \frac{U}{L} \right)^{-2n}, T \right)
\]

\[
\left( 'L-First-U-In Portfolio (LFUI), Part II' \right)
\]

The portfolio-weight factors are

\[
m_{BS}(0, \ln L, \ln U) = e^{-+\left( \ln L - \ln U \right)\left( -\frac{r-d-\frac{1}{2}\sigma^2}{\sigma^2} \right)} + \left| \ln L - \ln U \right| \left( -\frac{r-d-\frac{1}{2}\sigma^2}{\sigma^2} \right)
\]

and

\[
m_{BS}(0, \ln U, \ln L) = e^{-+\left( \ln U - \ln L \right)\left( -\frac{r-d-\frac{1}{2}\sigma^2}{\sigma^2} \right)} + \left| \ln U - \ln L \right| \left( -\frac{r-d-\frac{1}{2}\sigma^2}{\sigma^2} \right),
\]

where the constant \( \sigma (\sigma \geq 0) \) is the local volatility of the underlying logprice. \( m_{BS}(\lambda, x_0, b) \) is the moment generating function of the risk-adjusted logprice’s first exit time through some barrier \( b \) once it starts from the initial level \( x_0 \). \( \lambda (\lambda \geq 0) \) is the moment generating function parameter.
**Proof.** See the appendix. ■

Notice that, in absolute value, Part II of LFUI dominates Part I of UFLI. They have the same portfolio amounts, same strikes, but LFUI has higher down-in barriers than UFLI. On the other hand, Part I of LFUI is dominated in absolute value by Part II of UFLI. They have the same portfolio amounts, same strikes, but UFLI has lower up-in barriers than LFUI. Given that the original strike is within the double barrier corridor, Part II of UFLI actually consists of vanilla call options because its up-in barriers are lower than their corresponding strikes. Table I displays the structure of the FDBH.

Portfolio amounts and single barriers are fully characterized in terms of the risk-adjusted probability of the price ever travelling the distance \([L, U]\) from \(L\) to \(U\) and in the opposite direction, \(m_{BS}(0, \ln L, \ln U)\) and \(m_{BS}(0, \ln U, \ln L)\), respectively. Indeed, these two excursion probabilities make the portfolio weights. The factor \((\frac{U}{L})^{-1}\) rescales the single knockin option prices, their strikes, and their barriers. \((\frac{U}{L})^{-1}\) would be the risk-adjusted probability of the price ever travelling from \(L\) to \(U\) and in the opposite direction if the risk-adjusted price had zero local drift. Zero local drift for the underlying asset implies zero cost of carry and this is a natural assumption only for forward and futures contracts.

**Proposition 2** Under the Black-Scholes assumptions and with zero cost of carry, the FDBH and the static hedge proposed by CEG coincide.

**Proof.** See the appendix. ■

Table II illustrates the equivalence between the two hedges in the case of zero carrying costs. Since they correspond layer by layer, their hedging architecture is the same. CEG conveniently represents each UFLI knockin position with one non-barrier (less exotic) option position but represents each LFUI knockin position with three non-barrier option positions.

### A. Hedging architecture

If the upper (lower) barrier is hit first, the single barrier contract written on the lower (upper) barrier contributes positive unwanted value, \(C^L_{\text{knockin}}\) (\(C^U_{\text{knockin}}\)). Figure 1 quantifies such unwanted value. Much of the action happens at the lower barrier. Along there, for high log-price volatility levels (50%), the \(U\)-in call makes the Basic Portfolio value exceed the vanilla call value by nearly 100%.
Table I: The Full Double Barrier Hedge (FDBH)

$E^Q$ denotes expectation under the risk-adjusted probability measure and $T_U$ ($T_L$) is the first time the underlying price reaches the barrier $U$ ($L$). The arguments of the option price functions are the current underlying asset price $S_0$, the strike price $K$, and the time to maturity, $T$. $C_{knockin}^{L,U}$ denotes the price of a double knockin call with upper barrier $U$ and lower barrier $L$. $C_{knockin}^U$ denotes the price of a single knockin call with barrier $U$. $r$ is the risk-free rate and $d$ is the asset’s payout rate. $\sigma$ ($\sigma \geq 0$) is the local volatility of the underlying logprice ($r$, $d$, and $\sigma$ are constant).

$$
C_{knockin}^{L,U} (S_0, K, T) = \nonumber 
+ C_{knockin}^U (S_0, K, T) + C_{knockin}^L (S_0, K, T)
$$

Basic Portfolio

$$
-E^Q \left( e^{-rT} 1_{(T_U < T_L)} C_{knockin}^L (U, K, T - T_U) \mid S_0 \right) \nonumber 
= \nonumber 
\sum_{n=1}^{\infty} e^{-\left( \frac{\left( r - \frac{1}{2} \sigma^2 \right) + 1}{\frac{\sigma^2}{2}} \right) n (\ln U - \ln L)} \times C_{knockin}^L \left( S_0, K \left( \frac{U}{T} \right)^{-2n}, T \right)
$$

U-First-L-In (UFLI)

(single barrier ($L \left( \frac{U}{T} \right)^{-2n}$) down-and-in calls with barrier below the strike ($K \left( \frac{U}{T} \right)^{-2n}$))

$$
-E^Q \left( e^{-rT} 1_{(T_U < T_L)} C_{knockin}^U (L, K, T - T_U) \mid S_0 \right) \nonumber 
= \nonumber 
\sum_{n=1}^{\infty} e^{-\left( \frac{\left( r - \frac{1}{2} \sigma^2 \right) + 1}{\frac{\sigma^2}{2}} \right) n (\ln U - \ln L)} \times C_{knockin}^U \left( S_0, K \left( \frac{U}{T} \right)^{2n}, T \right)
$$

L-First-U-In (LFUI)

(single barrier ($L \left( \frac{U}{T} \right)^{2n}$) up-and-in calls with barrier below the strike ($K \left( \frac{U}{T} \right)^{2n}$) = standard calls)

$$
-E^Q \left( e^{-rT} 1_{(T_L < T_U)} C_{knockin}^U (L, K, T - T_L) \mid S_0 \right) \nonumber 
= \nonumber 
\sum_{n=1}^{\infty} e^{-\left( \frac{\left( r - \frac{1}{2} \sigma^2 \right) + 1}{\frac{\sigma^2}{2}} \right) n (\ln U - \ln L)} \times C_{knockin}^U \left( S_0, K \left( \frac{U}{T} \right)^{2n}, T \right)
$$

Basic Portfolio

(single barrier ($U \left( \frac{U}{T} \right)^{2n}$) up-and-in calls with barrier above the strike ($K \left( \frac{U}{T} \right)^{2n}$))

$$
-E^Q \left( e^{-rT} 1_{(T_L < T_U)} C_{knockin}^L (L, K, T - T_L) \mid S_0 \right) \nonumber 
= \nonumber 
\sum_{n=1}^{\infty} e^{-\left( \frac{\left( r - \frac{1}{2} \sigma^2 \right) + 1}{\frac{\sigma^2}{2}} \right) n (\ln U - \ln L)} \times C_{knockin}^L \left( S_0, K \left( \frac{U}{T} \right)^{-2n}, T \right)
$$

Basic Portfolio

(single barrier ($U \left( \frac{U}{T} \right)^{-2n}$) down-and-in calls with barrier above the strike ($K \left( \frac{U}{T} \right)^{-2n}$))
The arguments of the option price functions are the current underlying price, \( S_0 \), the strike price \( K \), and the time to maturity, \( T \). \( C_{\text{knockin}}^{L,U} \) denotes the price of a double knockin call with upper barrier \( U \) and lower barrier \( L \). \( C_{\text{knockin}}^Y \) denotes the price of a single knockin call with barrier \( Y \). \( C \) denotes the vanilla call price. \( P \) denotes the vanilla put price. \( BP \) (\( GP \)) is the price of a European binary (gap) put option, \( BC \) (\( GC \)) is the price of a European binary (gap) call option. The riskfree rate and the asset’s payout rate are equal so that the risk-adjusted drift of the underlying asset price is zero. The local volatility of the returns on the underlying asset can be time-dependent and price-dependent but must satisfy a symmetry-in-logprice condition. The volatility of the underlying asset price, \( \sigma \left( S_t \right) \), is a known function \( S_t \sigma \left( S_t, t \right) \, dt \) of the underlying price \( S_t \) at time \( t \) and it must satisfy the condition \( \sigma \left( S_t, t \right) = \sigma \left( \frac{S_t}{S_0}, t \right) \) for all \( S_t \geq 0 \) and \( t \) in \([0, T]\). The symmetry-in-logprice condition is trivially satisfied under the Black-Scholes assumptions.

\[
\begin{align*}
C_{\text{knockin}}^{L,U} \left( S_0, K, T \right) &= \\
&= \left( KU^{-1}C \left( S_0, K^{-1}U^2, T \right) + (U - K) \left( 2BC \left( S_0, U, T \right) + U^{-1}C \left( S_0, U, T \right) \right) \right) \\
&\quad \text{(single barrier (U) up-and-in call with barrier below the strike (K))} \\
&\quad + KL^{-1}P \left( S_0, K^{-1}L^2, T \right) \\
&\quad \text{(single barrier (L) down-and-in call with barrier below the strike (K))} \\
&\quad + \sum_{n=1}^{\infty} \left( \frac{U}{L} \right)^n \, KL^{-1}P \left( S_0, \left( \frac{L}{K} \right) L \left( \frac{U}{L} \right)^{-2n}, T \right) \\
&\quad \text{(single barrier \( (L(L^{2n})^{-2n}) \) down-and-in calls with barrier below the strike \( (K(L^{2n})^{-2n}) \))} \\
&\quad - \sum_{n=1}^{\infty} \left( \frac{U}{L} \right)^{-n} \, C \left( S_0, K \left( \frac{U}{L} \right)^{2n}, T \right) \\
&\quad \text{(single barrier \( (L(L^{2n})^{-2n}) \) up-and-in calls with barrier below the strike \( (K(L^{2n})^{-2n}) \))} \\
&\quad + \sum_{n=1}^{\infty} \left( \frac{U}{L} \right)^{-n} \, \frac{KU^{-1}C \left( S_0, \left( \frac{U}{L} \right)^{2n}, T \right)}{2e^{2nln \left( U - L \right)} + 2BC \left( S_0, U, \left( \frac{U}{L} \right)^{-2n}, T \right)} \\
&\quad \text{(single barrier \( (U(U^{2n})^{2n}) \) up-and-in calls with barrier above the strike \( (K(U^{2n})^{2n}) \))} \\
&\quad - \sum_{n=1}^{\infty} \left( \frac{U}{L} \right)^{+n} \, \frac{P \left( S_0, K \left( \frac{U}{L} \right)^{-2n}, T \right)}{U^{-12GP \left( S_0, U, \left( \frac{U}{L} \right)^{-2n}, T \right) + U^{-1}C \left( S_0, U, \left( \frac{U}{L} \right)^{-2n}, T \right)}} \\
&\quad \text{(single barrier \( (U(U^{2n})^{2n}) \) down-and-in calls with barrier above the strike \( (K(U^{2n})^{2n}) \))}
\end{align*}
\]
Figure 1: Unwanted value contribution of the Basic Portfolio along the two barriers. The $80-in and $100-in calls have strike $90 and 3-month maturity. The cost of carrying the underlying asset (riskfree rate, 6%, minus asset’s payout rate, 3%) is 3%.

The value of UFLI (Parts I and II) eliminates the unwanted value $C_{\text{knockin}}^L$ along the upper barrier. Indeed, UFLI is a short position in a $L$-in call that becomes available as soon as the upper barrier $U$ is hit first before or at maturity, with current cost equal to

$$-E^Q \left( e^{-rT_U} 1_{\{T_U<T_L\}} C_{\text{knockin}}^L (U, K, T - T_U) \mid S_0 \right).$$

$E^Q$ denotes expectation under the risk-adjusted probability measure and $T_U$ ($T_L$) is the first time the underlying price reaches the upper barrier $U$ ($L$). If the lower barrier is hit first, the indicator function calculated on the event $\{T_U<T_L\}$ is zero so that there is zero unwanted contribution there.

The value of LFUI (Parts I and II) offsets the unwanted value $C_{\text{knockin}}^U$ along the lower barrier. Indeed, LFUI is a short position in a $U$-in call that becomes available as soon as the lower barrier $L$ is hit first before or at maturity, with current cost equal to
\[-E^Q \left( e^{-rT_L} 1_{\{T_L < T_U\}} C_{\text{knockin}}^{\text{U}} (L, K, T - T_L) \mid S_0 \right) \cdot\]

If the upper barrier is hit first, the indicator function calculated on the event \(\{T_L < T_U\}\) is zero so that there is zero unwanted value contribution there.

How does the hedging architecture work layer by layer? Consider the replication of the \$80 / \$100 double barrier knockin call. One must zero out unwanted value along each barrier. For example, along \$100, the positive influence of the \$80-in call is offset by selling an amount

\[
\frac{m_{BS}(0, \ln 100, \ln 80)}{m_{BS}(0, \ln 80, \ln 100)} \times \left( \frac{100}{80} \right)^2 = 0.6243
\]

of up-in calls with barrier \(80 \left( \frac{100}{80} \right)^2 = 125\) and strike \(90 \left( \frac{100}{80} \right)^2 = 140.63\).

Along \$80, the positive influence of the \$100-in call is offset by selling an amount

\[
\frac{m_{BS}(0, \ln 80, \ln 100)}{m_{BS}(0, \ln 100, \ln 80)} \times \left( \frac{100}{80} \right)^2 = 1.6017
\]

of down-in calls with barrier \(100 \left( \frac{100}{80} \right)^2 = 64\) and strike \(90 \left( \frac{100}{80} \right)^2 = 57.60\). However, these short positions generate negative value along the opposite barrier so that other knockin positions must be added. Each additional position hedges at one barrier but creates an error at the other barrier. The size of that error decreases to zero with the number of hedging layers added. With a zero cost of carry, the barrier contracts that make up the FDBH layers can be exactly decomposed into portfolios of non-barrier contracts and the hedging architecture described in CEG, pp. 1174-1176, ensues.

3. Concluding remarks

Barrier derivatives are becoming increasingly liquid. Double barrier options provide investors and risk managers with cheaper means to place bets and to hedge their exposures respectively without paying for the price ranges that they believe unlikely to occur. Double barrier options stipulate a double barrier price corridor that encompasses the initial level of the underlying asset price and the options are triggered or terminated whenever the underlying asset price breaches either barrier for the first time before or at maturity. The mutual dependence of the two barriers makes these options difficult to price. I show that, under the Black-Scholes assumptions, the double barrier option price is a weighted sum of single barrier option prices. The mutual dependence of the
two barriers also makes these options difficult to hedge. My pricing representation implies a clear static hedging strategy (the FDBH). Double barrier hedges offer full protection only if unwound along the barriers and, there, the FDBH has automatic unwinding of its most important legs. This feature of the FDBH is aligned with Taleb’s (1998) risk management tip of avoiding hedging discontinuous exposures (barrier instruments) with continuous ones (non-barrier instruments).
APPENDIX

The underlying asset has cost of carry equal to \( r - d \) (\( r \) is the constant riskfree rate and \( d \) is the asset’s payout rate). Its risk-adjusted logprice, \( x_t = \ln S_t \), follows a diffusion process with dynamics:

\[
dx_t = (r - d(x_t)) \, dt - \frac{1}{2} \sigma^2(x_t) \, dt + \sigma(x_t) \, dW_t,
\]

where \( W_t \) is a Standard Brownian Motion and \( r, d, \) and \( \sigma \) are time-homogeneous and satisfy the conditions that allow for \( x_t \)'s existence and uniqueness. Set \( \ln L = b - \ln S_0 \), \( \ln U = b + (b - x_0 < b) \), and a finite time horizon (option’s maturity), \( T \).

The probability density of \( x_t \)'s transition from \( x_0 \) to \( x \) during \( T \), \( p(x_0, x, T) \) has time-horizon Laplace transform given by:

\[
L(\lambda, x_0, x) = \int_0^\infty \exp(-\lambda T) \, p(x_0, x, T) \, dT, \quad \lambda \geq 0.
\]

Taking time-horizon Laplace transforms simplifies the analysis. The Partial Differential Equation (PDE) dynamics of \( p(x_0, x, T) \) turns into an Ordinary Differential Equation (ODE) dynamics. A further simplification comes from that the Convolution Property of Laplace transforms. Because of \( x_t \)'s Strong Markov Property, a probability density involving first exit times until the time horizon can be written as a convolution of similar densities stopped at the time horizon. A transformed convolution is the product of the transformed densities involved in the convolution.

**Proposition 3** The time-horizon Laplace transform \( L(\lambda, x_0, x) \) satisfies the ODE

\[
\frac{1}{2} \sigma^2(x_0) L_{x_0}^2 + \left( \mu(x_0) - \frac{1}{2} \sigma^2(x_0) \right) L_{x_0} - \lambda L = 0, \quad \text{(Laplace ODE)}
\]

where \( L_{x_0} \) and \( L_{x_0}^2 \) denote \( L(\lambda, x_0, x) \)'s first and second derivatives with respect to \( x_0 \). \( L(\lambda, x_0, x) \) is positive and unique.

**Proof.** The probability density of \( x_t \)'s transition is an Itô process and it can be conceived as a conditional expectation, that is, as a local martingale. Thus, its local drift must be zero, which means that the expectation, conditional on \( x_0 \), of \( p \)'s infinitesimal changes is null, \( E(dp | x_0) = 0 \). This is \( p \)'s backward equation and one gets the Laplace ODE by taking time-horizon Laplace transforms in it. \( L \) is positive because \( p \) is non-negative in all its arguments and it is unique because of Laplace transforms’ uniqueness.

The moment generating function of \( x_t \)'s first exit time through some barrier \( b \),

\[
m(\lambda, x_0, b),
\]

is related to the Laplace transform of the probability density of \( x_t \)'s transition from \( x_0 \) to \( b \) as well as that of the probability density of \( x_t \)'s transition from \( b \) to the same level \( b \).

**Proposition 4** The moment generating function of \( x_t \)'s first exit time through an upper barrier \( b^+ \) (lower barrier \( b^- \)), \( m(\lambda, x_0, b^\pm) \), satisfies the Laplace ODE with these initial conditions:

\[
m(\lambda, b^\pm, b^\pm) = 1,
\]

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and, for \( \lambda > 0 \),

\[
0 < m(\lambda, x_0, b^\pm) < m(0, x_0, b^\pm) \leq 1. 
\]  

(‘Probability Bound’)

The solution to the Laplace ODE with those initial conditions is given by

\[
m(\lambda, x_0, b^\pm) = \frac{L(\lambda, x_0, b^\pm)}{L(\lambda, b^\pm, b^\pm)}. 
\]  

(Single Barrier M.G.F.)

The Single Barrier M.G.F.'s enjoy the following properties. The Single Barrier M.G.F. \( m(\lambda, x_0, b^-) \) is strictly decreasing in \( x_0 \) and the Single Barrier M.G.F. \( m(\lambda, x_0, b^+) \) is strictly increasing in \( x_0 \). For finite \( c \geq 0 \),

\[
m(\lambda, x_0, b^+ + c) = m(\lambda, x_0, b^+)m(\lambda, b^+, b^+ + c), 
\]  

(‘Strong Markov Up’)

\[
m(\lambda, x_0, b^- - c) = m(\lambda, x_0, b^-)m(\lambda, b^-, b^- - c), 
\]  

(‘Strong Markov Down’)

**Proof.** Let \( \tau_{b^\pm} \) be \( x_t \)'s first exit time through \( b^\pm \). \( \tau_{b^\pm} \)'s moment generating function satisfies the Laplace ODE as it is the Laplace transform of \( \tau_{b^\pm} \)'s probability density, which in turn satisfies the backward equation (its local drift is zero). If \( x_0 \to b^\pm \), the first exit time is zero for sure, that is, \( \exp(-\lambda \tau_{b^\pm}) \) is constant and equal to 1. This gives the first initial condition. The second initial condition, ‘Probability Bound’, comes from the fact that \( \exp(-\lambda \tau_{b^\pm}) \) times \( \tau_{b^\pm} \)'s probability density is not greater than \( \tau_{b^\pm} \)'s probability density and that \( m(0, x_0, b^\pm) \) is the probability of ever reaching the barrier \( b^\pm \). The result for Single Barrier M.G.F. follows from \( L(\lambda, x_0, x) \)'s structure and properties. The result also comes from Jamshidian (1997) who makes use of the Strong Markov Property and of the Convolution Property of Laplace transforms. The results for ‘Strong Markov Up’ and ‘Strong Markov Down’ follow from \( L(\lambda, x_0, x) \)'s structure and properties. The Strong Markov Property and of the Convolution Property of Laplace transforms prompt an alternative derivation of them. 

Let \( m^+ (\lambda, x_0, b^-, b^+) \) (\( m^- (\lambda, x_0, b^-, b^+) \)) be the moment generating function of \( x_t \)'s first exit time through the upper barrier \( b^+ \) (lower barrier \( b^- \)) whithout any passage through the lower barrier \( b^- \) (upper barrier \( b^+ \)). The sum of \( m^+ \) and \( m^- \) gives the moment generating function of \( x_t \)'s first exit time through either barrier. \( m^+ \) and \( m^- \) satisfy the Laplace ODE with these initial conditions:

\[
m^+ (\lambda, b^+, b^-, b^+) = 1, \quad m^+ (\lambda, b^-, b^-, b^+) = 0, 
\]  

\[
m^- (\lambda, b^+, b^-, b^+) = 0, \quad m^- (\lambda, b^-, b^-, b^+) = 1. 
\]  

This is because, if \( x_0 \to b^+ \), the upper barrier is reached for sure and from the very beginning, without touching the lower barrier \( b^- \). This implies \( m^+ = 1 \) and \( m^- = 0 \). The reverse holds for \( x_0 \to b^- \).

**Proposition 5** If \( x_t \) is an Arithmetic Brownian Motion (\( \mu \) and \( \sigma \) are constants), the moment generating functions \( m^\pm (\lambda, x_0, b^-, b^+) \) can be decomposed as follows:

\[
m^+ (\lambda, x_0, b^-, b^+) = 
\sum_{n=0}^{\infty} \left( \frac{m(0, b^+, b^-)}{m(0, b^-, b^+)} \right)^n m(\lambda, x_0, b^+ + 2n(b^+ - b^-)) - \sum_{n=0}^{\infty} \left( \frac{m(0, b^-, b^-)}{m(0, b^-, b^+)} \right)^{n+1} m(\lambda, x_0, b^+ - 2(n + 1)(b^+ - b^-)) 
\]  

\((m^+ \text{'s Form})\)

and

\[
m^- (\lambda, x_0, b^-, b^+) = 
\]  

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Brownian Motion hypothesis yields lead to the \(m\)'s Form. The operator that generates the Laplace ODE is linear so that a linear combination of Single Barrier M.G.F.s satisfies it. ‘Probability Bound’ implies that the product of \(m(\lambda, x_0, b^-)\) and \(m(\lambda, x_0, b^+)\) is linear so that a linear combination of Single Barrier M.G.F.s satisfies it. 'Probability Bound' implies \(m\)'s Form. The operator that generates the Laplace ODE is linear so that a linear combination of Single Barrier M.G.F.s satisfies it. ‘Probability Bound’ implies that the product of \(m(\lambda, x_0, b^-)\) and \(m(\lambda, x_0, b^+)\) is less than 1. Thus, the following linear combination of \(m(\lambda, x_0, b^+)\) and \(m(\lambda, x_0, b^-)\),

\[
\sum_{n=0}^{\infty} \left( \frac{m(0, b^+, b^-)}{m(0, b^-, b^-)} \right)^n m(\lambda, x_0, b^- - 2n(b^+ - b^-)) - \sum_{n=0}^{\infty} \left( \frac{m(0, b^+, b^-)}{m(0, b^-, b^-)} \right)^{n+1} m(\lambda, x_0, b^- + 2(n+1)(b^+ - b^-)).
\]

\(m\)'s Form)

**Proof.** I focus on the \(m^+\)'s Form. Similar arguments justify the \(m^-\)'s Form. The operator that generates the Laplace ODE is linear so that a linear combination of Single Barrier M.G.F.s satisfies it. ‘Probability Bound’ implies that the product of \(m(\lambda, x_0, b^-)\) and \(m(\lambda, x_0, b^+)\) is less than 1. Thus, the following linear combination of \(m(\lambda, x_0, b^+)\) and \(m(\lambda, x_0, b^-)\),

\[
m(\lambda, x_0, b^+) \sum_{n=0}^{\infty} \left( m(\lambda, b^+, b^-) m(\lambda, b^-, b^+) \right)^n - m(\lambda, x_0, b^-) \sum_{n=0}^{\infty} \left( m(\lambda, b^-, b^+) m(\lambda, b^-, b^-) \right)^n m(\lambda, b^-, b^+),
\]

satisfies the Laplace ODE and meets \(m^+\)'s two initial conditions. The same preliminary decomposition can be obtained from Jamshidian’s (1997) analysis by expanding \(\left( 1 - \frac{L(\lambda, b^-, b^+)}{L(\lambda, b^-, b^-)} \right)^{-1}\) in power series. The Arithmetic Brownian Motion hypothesis yields

\[
m(\lambda, b^+, b^-) = m(\lambda, b^-, b^+) \frac{m(0, b^+, b^-)}{m(0, b^-, b^-)}.
\]

The Arithmetic Brownian Motion hypothesis implies that the travel distance \([b^-, b^+]\) can be shifted by any shifting factor \(\pm c\). Set \(c\) equal to either \(n(b^+ - b^-)\) or \(\frac{1}{2}n(b^+ - b^-)\). Then, ‘Strong Markov Up’ and ‘Strong Markov Down’ lead to the \(m^+\)'s Form. □

**Proof of Proposition 1**

The probability density which prices the double knockin contracts has the following option-maturity Laplace transform:

\[
m^+(\lambda, \ln S_0, \ln L, \ln U) \cdot L(\lambda, \ln U, \ln S_T) + m^-(\lambda, \ln S_0, \ln L, \ln U) \cdot L(\lambda, \ln L, \ln S_T).
\]

Proposition 5 as well as option prices’ homogeneity of degree 1 in the initial price, the strike, and the possible barriers, can be used. This gives the FDBH result and completes the proof. □

**Proof of Proposition 2**

The Put Call Symmetry (PCS) states that, under the Black-Scholes assumptions with zero risk-adjusted drift of the underlying price, the value of an amount

\[
\frac{1}{\sqrt{\text{call strike}}}
\]

of calls is equal to the value of an amount

\[
\frac{1}{\sqrt{\text{put strike}}}
\]

of puts, if the geometric mean of the call strike and the put strike is the current underlying price:

\[
\sqrt{\text{call strike}} \times \sqrt{\text{put strike}} = S_0.
\]

By means of the PCS, Bowie and Carr (1994) and Carr, Ellis, and Gupta (1998) show that a European single barrier option can be replicated by a portfolio of European vanilla calls, European vanilla puts, and European binary options. A European binary call (put) is a cash-or-nothing option which pays $1 if the underlying price is above
(below) the strike price, and zero otherwise. In particular, they prove the following results for down-in call options, and up-in call options respectively:

\[
\begin{align*}
C_{\text{knock-in}}^L(S_0, K, T) &= KL^{-1}P(S_0, K^{-1}L^2, T), \quad L < K, \\
C_{\text{knock-in}}^U(S_0, K, T) &= P(S_0, K, T) + (H - K) \left(\frac{2BP(S_0, L, T)}{L^{-1}P(S_0, L, T)}\right), \quad L > K, \\
C_{\text{knock-in}}^U(S_0, K, T) &= KU^{-1}C(S_0, K^{-1}U^2, T) + (U - K) \left(\frac{2BC(S_0, U, T)}{U^{-1}C(S_0, U, T)}\right), \quad U > K,
\end{align*}
\]

where \(P(S_0, K, T)\) is the price of a European vanilla put option with strike \(K\) and maturity \(T\), \(BP(S_0, K, T)\) is the price of a European binary put option, and \(BC(S_0, K, T)\) is the price of a European binary call option. PCS also links European gap put options to European binary put options:

\[
GP(S_0, K, T) = K \cdot BP(S_0, K, T) - P(S_0, K, T),
\]

where \(GP(S_0, K, T)\) is a European gap put option. A European gap call (put) is a cash-or-nothing option which pays a dollar amount equal to the underlying price if the underlying price is above (below) the strike price, and zero otherwise.

Breaking down the FDBH single barrier components by means of the PCS results and setting the cost of carry, \(r - d\), to zero achieves the element-by-element equivalence. \(\blacksquare\)
REFERENCES


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