Investment under higher uncertainty when business conditions worsen

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Much of the work on investment under uncertainty assumes that the project’s value follows Geometric Brownian Motion (GBM) with constant volatility. I use a more general assumption for the project-value process. I use the so-called Constant-Elasticity-of-Variance (CEV) diffusion model where the volatility is a non-increasing function of the project’s value. I show that, if the CEV volatility structure holds, the firm that uses the standard GBM assumption is exposed to significant errors when making the optimal investment decision. In particular, the GBM assumption often yields suboptimal postponement of investment and undervalues the option of waiting to invest under deteriorated business conditions.

JEL-Classification: G12, G13, G31.

Keywords: Investment under uncertainty; Real options; Diffusion processes; CEV model.

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Much of the work on investment under uncertainty assumes that the project’s value follows Geometric Brownian Motion (GBM) with constant volatility. I use a more general assumption for the project-value process. I use the so-called Constant-Elasticity-of-Variance (CEV) diffusion model where the volatility is a non-increasing function of the project’s value. I show that, if the CEV volatility structure holds, the firm that uses the standard GBM assumption is exposed to significant errors when making the optimal investment decision. In particular, the GBM assumption often yields suboptimal postponement of investment and undervalues the option of waiting to invest under deteriorated business conditions.

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1 Introduction

I extend the classic analysis of investment under uncertainty in McDonald and Siegel (1986) - as reviewed in Dixit and Pindyck (1994, pp. 136-161) - to a real options setting where the firm observes more volatility in the project’s value as business conditions deteriorate. I do this by modelling the project’s value as a Constant-Elasticity-of-Variance (CEV) diffusion process and by employing within a real options setting closed-form results from the CEV-based work of Davydov and Linetsky (2001) on financial path-dependent options.

My CEV-based analysis relaxes the strong assumption of constant local volatility in the analysis of McDonald and Siegel (1986) who take the project’s value to follow Geometric Brownian Motion (GBM). I make the more general assumption that the local volatility of the project’s value is a non-increasing function of the project’s value itself. This extension of the project-value-volatility model specification better fits projects whose value is spanned by a trading strategy that is long in the stock market portfolio. Higher volatility in downturns is a well-known stylized fact for stock markets and goes beyond the mere leverage effect as it appears to be driven by crashophobia - see, for example, Jackwerth and Rubinstein (1998).

The present paper gives a two-fold contribution. First, I employ within a real options setting the closed-form expressions for the solutions of the stationary Black-Scholes Differential Equation (DE) with CEV state-dependent volatility. Second, I use the closed-form option pricing formulae to carry out a comparative statics analysis. I show that the investment decision under the CEV process can deviate substantially from the GBM-based
investment decision. In particular, if the GBM-based constant volatility is well specified when the project’s value yields a zero Net Present Value (NPV), the CEV-based optimal investment rule will wait much less to invest than the GBM-based rule will do. For deteriorated business conditions the GBM-based additional delay in investing does not mean a greater value of the option of waiting to invest. CEV volatility is quite high in such a region of business conditions so that CEV option values are higher than GBM option values.

These findings are in line with the results in Davydov and Linetsky (2001). The option of waiting to invest is an American option. American options are path-dependent options. Indeed, they are barrier options that enable the option holder to select the option-value-maximizing barrier level. Davydov and Linetsky (2001) demonstrate that the prices of options, which depend on extrema, such as barrier options, can be quite sensitive to the specification of the underlying price process and show that the option dealer who uses the standard GBM assumption is exposed to significant pricing errors when handling path-dependent options.

Higher CEV volatility in deteriorating business conditions yields a higher chance of the project-demise event - that is, project-value absorption at the zero level. This adds interest in the analysis for two reasons: (i) The GBM setting in McDonald and Siegel (1986) assigns zero probability mass to the project-demise event; (ii) As opposed to a setting where the project suffers of unpredictable Poisson-jump demise, the predictable event of the project’s demise cannot be dealt with by just increasing the discount rate.

The work is organized as follows. Section 2 introduces the investment problem under
CEV uncertainty. Section 3 looks at its dynamic programming formulation. Section 4 solves the problem by contingent claims analysis. Section 5 discusses the characteristics of the optimal investment rule. Section 6 concludes.

2 Investment under CEV uncertainty

For a project with present value $V$, I consider the following problem: At what critical level $V^\ast$ of the project’s value is it optimal to pay a sunk cost $I$ in return for the project’s value itself? I focus only on projects for which stochastic changes in $V$ are spanned by a trading strategy that is long in the stock market portfolio - this can well happen in a standard Capital Asset Pricing Model (CAPM) economy. Such a spanning assumption justifies injecting the observed asymmetric volatility of the stock market into the project’s value and also makes the analysis akin to the treatment of perpetual American options. $V$ evolves according to the following CEV process (under the equivalent martingale measure $Q$):

\[ \frac{dV}{V} = (r - \delta) dt + \eta V^\theta dz, \quad (Q\text{-dynamics for } V) \]

where $r$ is the riskfree rate, $\delta$ is the payout rate, $\eta$ is the scale parameter for the local volatility of the project’s value, and $dz$ is the increment of a Wiener process. The CEV process takes its name from the fact that the elasticity of its local volatility with respect to the level of the process is constant and equal to $\theta$:

\[ V \frac{\partial}{\partial V} \ln (\eta V^\theta) = \theta. \]

I assume that

\[ \theta \leq 0, \]

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so that project-value volatility can increase as the project’s value decreases. Given the underlying CEV uncertainty, the probability of the project’s value being ever absorbed at zero, that is, the chance of the project’s demise, will never be trivial as long as the tighter parameter restriction $\theta < 0$ is imposed. This holds true with and without risk adjustment (cf. Cox (1975) and Davydov and Linetsky (2001), p. 952).

The value of the option to invest is $F(V)$. It is the opportunity cost of investing now rather than waiting.

3 The dynamic programming problem

The Bellman equation for the optimal investment problem is

$$F(V) = \max \left\{ \max \{V - I, 0\}, \exp(-rdt) \mathcal{E}[F(V) + dF \mid V; \theta] \right\},$$

with conditions,

$$F(V^*) = V^* - I,$$  
(Value Matching Condition)

$$\frac{d}{dV} F|_{V=V^*} = \frac{d}{dV} (V - I)|_{V=V^*} = 1,$$  
(Smooth Pasting Condition)

$$F(0) = 0,$$  
(Demise Condition)

where $\mathcal{E}[\cdot \mid V; \theta]$ is the conditional expectation under the equivalent martingale measure $Q$ given the project’s present value $V$. 
4 Solution by contingent claims analysis

In the absence of any asymmetry in the project-value volatility ($\theta = 0$), the firm is back to the benchmark GBM case. Consistency of notation with the classic review of Dixit and Pindyck (1994, pp. 136-161) makes convenient representing the case $\theta = 0$ with the following motion for the project’s value:

$$\frac{dV}{V} = (r - \delta)\, dt + \sigma dz. \quad (Q\text{-dynamics for } V \text{ with } \theta = 0)$$

The benchmark critical value $V^*, b$ is

$$V^*, b = \frac{\beta_1}{\beta_1 - 1} I,$$

$$\beta_1 = \frac{1}{2} - \frac{(r - \delta)}{\sigma^2} + \sqrt{\left[\frac{(r - \delta)}{\sigma^2} - \frac{1}{2}\right]^2 + \frac{2r}{\sigma^2} > 1},$$

and, given the stopping time $T_{V^*, b}$ at the benchmark critical value $V^*, b$, the firm’s option to invest has the benchmark value

$$F_b (V) = \mathcal{E} \left[ e^{-rT_{V^*, b}} \mid V; \theta = 0 \right] \left( V^*, b - I \right) 1_{\{V \leq V^*, b\}} + (V - I) 1_{\{V^*, b < V\}},$$

$$\mathcal{E} \left[ e^{-rT_{V^*, b}} \mid V; \theta = 0 \right] = \frac{\psi (V; \theta = 0)}{\psi (V^*, b; \theta = 0)},$$

$$\psi (V; \theta = 0) = (V)^{\beta_1}.$$

The function $\mathcal{E} \left[ e^{-rT_B} \mid V; \theta \right]$ is the value of a perpetual single barrier cash-at-hit claim written on the project’s value: One unit of numeraire is paid as soon as the project’s value
hits the barrier $B$. Since the project’s value is spanned, the values of claims written on it are also spanned.

For $\theta < 0$, there is no immediate investment in the region $\{V : V \in [0, V^*)\}$. Thus, the Bellman equation boils down to the Black-Scholes DE:

$$\mathcal{E} [dF | V; \theta] = F(V) r dt.$$ 

The value of any spanned contingent claim satisfies the Black-Scholes DE: In risk-adjusted expectation, the percentage increase in the spanned claim value must equal the claim’s percentage cost of carry. The boundary conditions introduce the free upper boundary $V^*$ as well as the absorbing barrier 0 so that the firm is facing a double barrier problem. Thus, the candidate solution must be the value of a perpetual double barrier cash-at-hit claim. The value of such a claim is (see, for example, Davydov and Linetsky (2001) and Going-Jaeschke and Yor (1999)):

$$\mathcal{E} \left[ e^{-rT} \mathbf{1}_{\{V_{T < T_0}\}} | V; \theta \right] = \frac{\psi(V; \theta)}{\psi(V^*; \theta)},$$

$$\psi(V; \theta) = \begin{cases} V^{\theta + \frac{1}{2}} \exp \left[ \frac{r}{2} y(V; \theta) \right] M_{k, \frac{1}{2}, \nu} [y(V; \theta)], & r - \delta \neq 0, \quad \theta < 0, \\ V^{\frac{1}{2}} I_{\nu} \left[ \sqrt{2r} h(V; \theta) \right] & r - \delta = 0, \quad \theta < 0, \end{cases}$$
\[ y(V; \theta) = \frac{|r - \delta|}{\eta^2 |\theta|} V^{-2\theta}, \]
\[ \epsilon = \text{sign} \left[ (r - \delta) \theta \right], \]
\[ k = \epsilon \left( \frac{1}{2} + \frac{1}{4\theta} \right) - \frac{r}{2 |(r - \delta) \theta|}, \]
\[ \nu = \frac{1}{2 |\theta|}, \]
\[ h(V; \theta) = \frac{1}{\eta |\theta|} V^{-\theta}, \]

where \( M_{k,m}[\cdot] \) is the Whittaker function\(^1\) and and \( I_\nu [\cdot] \) is the modified Bessel function of the first kind of order \( \nu \). The solution \( F(V) \),
\[ F(V) = \frac{\psi(V; \theta)}{\psi(V^*; \theta)} (V^* - I) 1_{\{V \leq V^*\}} + (V - I) 1_{\{V^* < V\}}, \]

has a form that satisfies by construction the Black-Scholes DE, the Value Matching Condition at the free boundary \( V^* \), and the Demise Condition at the absorbing barrier 0. The solution’s form is not constructed to fulfill the Smooth Pasting Condition. I force it to do so. This generates an analytic restriction, out of which the critical value \( V^* \) can be numerically obtained\(^2\):
\[ \frac{1}{\psi(V^*; \theta)} \left( \frac{d}{dV} \psi(V; \theta) \right) \bigg|_{V = V^*} = 1. \]

\(^1\)See Abramowitz and Stegun (1972, p. 505). The Whittaker function can be easily constructed via the confluent hypergeometric functions already built in the Mathematica software (cf. Davydov and Linetsky (2001, p. 956)).

\(^2\)Such a numerical computation is easily performed in the Mathematica software because of \( \psi(V; \theta) \)'s analytic nature.
5 Characteristics of the optimal investment rule

I here examine the characteristics of the optimal investment rule and the value of the investment opportunity. In picking the parameter values, I rely on the classic calibration in Dixit and Pindyck (1994, p. 153). Unless otherwise noted, in what follows I set the cost of the investment, $I$, equal to 1, $r = 4\%$, $\delta = 4\%$, and $\sigma = 20\%$ (at annual rates). Given these parameter values, the benchmark critical value $V^*_{b}$ is equal to $2I = 2$. The project’s value $V$ must be $100\%$ greater than the cost $I$ before the firm should invest. Jackwerth and Rubinstein (1998) find that typical values of the CEV elasticity implicit in the S&P 500 stock index option prices are strongly negative and are as low as $-4$. Unless otherwise noted, an adjustment for leverage in the equity index motivates the choice of $\theta = -2$ for the project-value CEV elasticity. I pin the CEV local volatility curve $\eta V^\theta$ at the GBM volatility level $\sigma (= 20\%)$ for the zero-NPV level $V = I = 1$ of the project’s value so that the scale parameter $\eta$ will be obtained from the restriction:

$$\eta V^\theta = \sigma \quad \text{for } V = I = 1.$$  

Figures 1, 3, 4, and 5 show that the CEV critical value never overshoots the GBM critical value given the zero-NPV pin $V = I = 1$ of the local volatility curve. Figure 5 shows that, for $\theta = -3$, the firm should invest when the project’s value is only $50\%$ greater than the investment cost $I$. CEV-based investment is anticipated because CEV uncertainty is decreasing in the project-value area that lays at the right-hand side of the cost $I$. The CEV critical value tends to equal the GBM critical value when the percentage benefit of carrying the project, $\delta - r$, swells up and when the CEV elasticity becomes trivial (that is, the local volatility flattens to the constant GBM level).
Figures 1 and 2 show the effect of setting the volatility pin $\sigma$ (the project-value pin $V$) of the local volatility curve $\eta V^\theta$ to the levels 5%, 10%, 20%, and 40% ($0.75$, $1.25$, $1.50$, and $2.00 = V^{*,b}$). The CEV critical value leaves behind the GBM critical value $V^{*,b}$ in the extreme situation when the CEV volatility is greater than the benchmark GBM volatility $\sigma = 20\%$ in the whole region $\{ V : V \in [0, V^{*,b}] \}$.

All figures show that low values of the project generate CEV values of the option of waiting to invest that are higher than the corresponding GBM option values. Indeed, those low levels of $V$ are associated to substantially high CEV volatility. The CEV probability of the project-demise event also soars in that project-value region and curbs the action of the enhanced local volatility until demise dominates over uncertainty to yield $F(0) = 0$.

6 Conclusions

I study the optimal investment rule under a CEV process for the project’s value. The CEV model with negative elasticity exhibits convex and monotonically decreasing volatility for the project’s value. If the project’s value is spanned by a trading strategy that is long the stock market portfolio, the CEV model improves the fit to the empirical observations because it heals the problem of misspecifying the project-value volatility, that is, the problem of assuming it constant. I show that, if the CEV volatility structure holds, the assumption of constant volatility can lead the firm to postpone investment way too much as well as to undervalue the option of waiting to invest under deteriorated business conditions.
References

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Figure 1: The value $F(V)$ of the option to invest vs. the project’s value $V$: The effect of the volatility pin $\sigma$ of the CEV local volatility curve $\eta V^\theta$

The solid line denotes the option value when $V$ follows a Constant-Elasticity-of-Variance (CEV) process: $\frac{dV}{V} = (r - \delta) dt + \eta V^\theta dz$ ($Q$-dynamics with $\theta < 0$). The dashed line denotes the option value when $V$ follows a Geometric Brownian Motion (GBM) process (the limit of the CEV process as $\theta \to 0$): $\frac{dV}{V} = (r - \delta) dt + \sigma dz$ ($Q$-dynamics with $\theta = 0$). The cost of the investment is $I = 1$, the riskfree rate is $r = 4\%$ and the project’s payout rate is $\delta = 4\%$. The CEV elasticity is $\theta = -2$. Given the GBM volatility $\sigma$ ($\sigma = 5\%, 10\%, 20\%, \text{and} 40\%$), the scale parameter $\eta$ is obtained from the following restriction: $\eta V^\theta = \sigma$ for the project-value pin $V = 1$. The ‘critical points’ denote the critical project-value levels that trigger immediate investment and the corresponding option values.
The value $F(V)$ of the option to invest vs. the project’s value $V$: The effect of the project-value pin of the CEV local volatility curve $\eta V^\theta$

The solid line denotes the option value when $V$ follows a Constant-Elasticity-of-Variance (CEV) process: $\frac{dV}{V} = (r - \delta) dt + \eta V^\theta dz$ (Q-dynamics with $\theta < 0$). The dashed line denotes the option value when $V$ follows a Geometric Brownian Motion (GBM) process (the limit of the CEV process as $\theta \to 0$): $\frac{dV}{V} = (r - \delta) dt + \sigma dz$ (Q-dynamics with $\theta = 0$). The cost of the investment is $I = 1$, the riskfree rate is $r = 4\%$ and the project’s payout rate is $\delta = 4\%$. The CEV elasticity is $\theta = -2$. Given the GBM volatility $\sigma = 20\%$, the scale parameter $\eta$ is obtained from the following restriction: $\eta V^{\theta} = \sigma$ for the project-value pin $V = 0.75, 1.25, 1.50, \text{ and } 2.00 (= V^{*,b}$, the benchmark GBM critical value). The ‘CEV’ and ‘GBM’ labels denote the critical project-value levels that trigger immediate investment and the corresponding option values.

Figure 2: The value $F(V)$ of the option to invest vs. the project’s value $V$: The effect of the project-value pin of the CEV local volatility curve $\eta V^\theta$
Figure 3: The value $F(V)$ of the option to invest vs. the project’s value $V$: The effect of the riskfree rate $r$

The solid line denotes the option value when $V$ follows a Constant-Elasticity-of-Variance (CEV) process: $\frac{dV}{V} = (r - \delta) dt + \eta V^\theta dz$ ($Q$-dynamics with $\theta < 0$). The dashed line denotes the option value when $V$ follows a Geometric Brownian Motion (GBM) process (the limit of the CEV process as $\theta \to 0$): $\frac{dV}{V} = (r - \delta) dt + \sigma dz$ ($Q$-dynamics with $\theta = 0$). The cost of the investment is $I = 1$. The riskfree rate $r$ covers the values 0.1%, 4%, 16%, and 50%. The project’s payout rate is $\delta = 4\%$. The CEV elasticity is $\theta = -2$. Given the GBM volatility $\sigma = 20\%$, the scale parameter $\eta$ is obtained from the following restriction: $\eta V^\theta = \sigma$ for the project-value pin $V = 1$. The ‘CEV’ and ‘GBM’ labels denote the critical project-value levels that trigger immediate investment and the corresponding option values.
Figure 4: The value $F(V)$ of the option to invest vs. the project’s value $V$: The effect of the project’s payout rate $\delta$

The solid line denotes the option value when $V$ follows a Constant-Elasticity-of-Variance (CEV) process: $\frac{dV}{V} = (r - \delta) \, dt + \eta V^\theta \, dz$ ($Q$-dynamics with $\theta < 0$). The dashed line denotes the option value when $V$ follows a Geometric Brownian Motion (GBM) process (the limit of the CEV process as $\theta \to 0$): $\frac{dV}{V} = (r - \delta) \, dt + \sigma \, dz$ ($Q$-dynamics with $\theta = 0$). The cost of the investment is $I = 1$. The riskfree rate is $r = 4\%$. The project’s payout rate $\delta$ covers the values $0.1\%$, $4\%$, $16\%$, and $50\%$. The CEV elasticity is $\theta = -2$. Given the GBM volatility $\sigma = 20\%$, the scale parameter $\eta$ is obtained from the following restriction: $\eta V^\theta = \sigma$ for the project-value pin $V = 1$. The ‘CEV’ and ‘GBM’ labels denote the critical project-value levels that trigger immediate investment and the corresponding option values.
Figure 5: The value $F(V)$ of the option to invest vs. the project’s value $V$: The effect of the elasticity parameter $\theta$, $\theta = V \frac{\partial}{\partial V} \ln (\eta V^\theta)$

The solid line denotes the option value when $V$ follows a Constant-Elasticity-of-Variance (CEV) process: $\frac{dV}{V} = (r - \delta) dt + \eta V^\theta dz$ (Q-dynamics with $\theta < 0$). The dashed line denotes the option value when $V$ follows a Geometric Brownian Motion (GBM) process (the limit of the CEV process as $\theta \to 0$): $\frac{dV}{V} = (r - \delta) dt + \sigma dz$ (Q-dynamics with $\theta = 0$). The cost of the investment is $I = 1$, the riskfree rate is $r = 4\%$ and the project’s payout rate is $\delta = 4\%$. The CEV elasticity $\theta$ covers the values $-0.25$, $-1$, $-2$, and $-3$. Given the GBM volatility $\sigma = 20\%$, the scale parameter $\eta$ is obtained from the following restriction: $\eta V^\theta = \sigma$ for the project-value pin $V = 1$. The ‘CEV’ and ‘GBM’ labels denote the critical project-value levels that trigger immediate investment and the corresponding option values.