Closed-form pricing of Benchmark Equity Default Swaps under the CEV assumption

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Abstract

Equity Default Swaps are new equity derivatives designed as a product for credit investors. Equipped with a novel pricing result, we provide closed-form values that give an analytic contribution to the viability of cross-asset trading related to credit risk.

*JEL-Classification:* G12, G33.

*Keywords:* Cross-Asset Trading of Credit Risk, Constant-Elasticity-of-Variance (CEV) Diffusion.
1 Introduction

Following rapid growth in the equity and credit derivatives markets, cross-asset products, which combine elements of credit and equity, have become more prominent. One such product is the Equity Default Swap (EDS) and it presents a challenge in terms of pricing – how to incorporate credit events into pricing models for equity-based instruments. EDSs are similar to Credit Default Swaps (CDS) insofar as a protection buyer makes a regular fee payment at intervals until either a trigger event or the contract maturity and receives from a protection seller a protection payment on the happening of the trigger event. The difference is in how the trigger event is determined. In a CDS, the trigger event is the occurrence of a credit event with respect to the reference entity. In an EDS, the trigger event is a fall in the share price of the reference entity to below a certain percentage of the price level at the inception of the trade. Since their first appearance in May 2003\(^1\), EDSs have been growing popular. EDSs can be used in yield-enhancement strategies (implemented by selling protection on reference entities that combine high equity volatility with a good credit rating) and as an alternative market access tool (counterparties that face limits on their exposure through CDSs have EDSs as an alternative method of trading credit risk). EDSs are

\(^1\)Wolcott (2004), p. 24, writes "Since the launch of EDSs last May, JP Morgan claims to have executed over $1 billion in notional." See also Sawyer (2003) and Sawyer (2004).
also viewed as attractive alternatives to CDSs in the context of synthetic Collateralized Debt Obligations for two reasons. First, the risk of a trigger event occurring on an EDS is more transparent. Second, determining the protection payment for an EDS is more certain since the EDS recovery rate is typically fixed at 50% of the notional amount.

We focus on the ‘Benchmark EDS’. We define it as an EDS contract with a trigger event corresponding to a 100% drop in share price since the commencement of the trade-share price absorption at zero. Default as share price absorption at zero is consistent with corporate finance theory and its clear equity-based definition renders valuation easy to implement\(^2\). Thus, we think the 100%-drop event in the equity market as an identifiable subset of the more opaque credit event that triggers the protection payment in a CDS.

The Geometric Brownian Motion (GBM) assumption is clearly mismated with the Benchmark EDS pricing task and we value the contract by means of assuming that the share price follows a Constant-Elasticity-of-Variance (CEV) diffusion, which brings in for free a well-known closed form of the probability of the 100%-drop event. We derive in closed form the truncated Laplace transform of the probability density function (p.d.f.) of the first hitting time of the CEV process at the zero level, which can serve as the

\(^2\)Structural models of EDS pricing (see Medova and Smith (2004)) also have corporate finance foundations. For such models, viability may be an issue—not all corporate liabilities are always tradable and leverage-ratio information is not always reliable.
Present Value (PV) of a Benchmark EDS protection payment. This result is, to the best of our knowledge, novel in the CEV-based asset-pricing literature\(^3\) and it naturally lends itself to credit derivatives pricing applications that enable cross-asset trading of credit risk. Our CEV approach comes along with parsimonious pricing flexibility. In particular, the closed-form CEV probability of default enables easy parameter calibration to implied risk-neutral probabilities of default. Among other models, Albanese and Chen (2004) also use the CEV model in the context of an EDS/CDS pricing study. They focus on the numerical assessment of the ratio of EDS rates to CDS rates rather than on CEV-based analytic pricing.

The rest of the work is organized as follows. Section 2 discusses the CEV assumption. Section 3 provides the pricing results. After the Conclusions (Section 4), an Appendix gathers the technical proofs.

2 The CEV assumption

The reference entity’s share has current price $S$ and we assume that, under the equivalent martingale measure $Q$, the share price process is a Constant-Elasticity-of-Variance (CEV) diffusion:

$$dS = (r - q) S dt + \sigma S^\rho dz,$$

where $r$ is the constant riskfree rate, $q$ is the constant dividend yield\(^4\), $\sigma$ is a constant scale factor for the instantaneous volatility, and $dz$ is the increment of a Wiener process under $Q$. The CEV process takes its name from the fact that the elasticity of the instantaneous volatility $\sigma S^{\rho - 1}$ with respect to the level of the process is constant and equal to $\rho - 1$:

$$S \frac{\partial}{\partial S} \ln (\sigma S^{\rho - 1}) = \rho - 1.$$

In line with much empirical evidence, we assume

$$\rho - 1 < 0$$

so that an inverse relationship between volatility and share price arises.

3 Pricing the Benchmark EDS

Given the maturity $T > 0$ of the Benchmark EDS contract and a 1$ notional amount, we want to calculate the no-arbitrage PV of the Benchmark EDS\(^4\)We consider the case $r - q > 0$. For stocks, the cost of carry is typically positive.
protection payment,

\[ E^Q \left[ \exp \left( -r \tau_0 \right) 1_{\{\tau_0 \leq T\}} \cdot 50\% \mid S \right] , \]

where \( \tau_0 \) is the first hitting time of the CEV process at the zero level, \( \tau_0 := \inf \{ s : S_s = 0 \} \). The object of interest is the truncated Laplace transform of \( \tau_0 \)'s p.d.f. with Laplace parameter \( \lambda \) set at the riskfree rate level \( (\lambda = r) \) and its closed-form expression is stated in the following proposition\(^5\) (the proof

\(^5\)See Davydov and Linetsky (2001) and Sbuelz (2004) for CEV-based non-truncated Laplace transform results. In particular, Davidov and Linetsky (2001), see pp. 953 and 956, point out that the \( T \)-truncated Laplace transform of \( \tau_0 \)'s \( Q \)-p.d.f. with Laplace parameter \( \lambda \) can be obtained by numerically inverting the closed-form non-truncated Laplace transform

\[ \frac{1}{a} E^Q_0 \left[ \exp \left( - (\lambda + a) \tau_0 \right) \right] , \]

where the inversion parameter is \( a > 0 \).
is in the Appendix). This notation backs the proposition:

\[ x = S^{2(1-\rho)}, \]

\[ A = \frac{2(r-q)}{\sigma^2(1-\rho)}, \]

\[ B = \frac{\lambda}{2(r-q)(1-\rho)}, \]

\[ \nu = \frac{1}{2(1-\rho)}, \]

\[ K = \frac{\sigma^2(1-\rho)}{2(r-q)} \left( 1 - e^{-2T(r-q)(1-\rho)} \right), \]

\[ H = \frac{(r-q)S^{2(1-\rho)}}{\sigma^2(1-\rho) \left[ 1 - e^{-2(r-q)(1-\rho)T} \right]^2}. \]

**Proposition 1** Under the CEV assumption, the truncated Laplace transform of \( \tau_0 \)'s p.d.f. with Laplace parameter \( \lambda \) (\( \geq 0 \)) admits this closed-form
expression:

\[
E^Q \left[ \exp(-\lambda \tau_0) 1_{\{\tau_0 \leq T\}} | S \right] = \lim_{\epsilon \downarrow 0} \sum_{n=0}^{\infty} a_n (A, B) \left( \frac{x}{2} \right)^n \frac{\Gamma(\nu - n, \frac{x}{2K}, \frac{x}{2})}{\Gamma(\nu)} ,
\]

\[
\Gamma(\nu) = \int_{0}^{+\infty} u^{\nu-1}e^{-u}du, \quad \text{(Gamma Function)}
\]

\[
\Gamma(\nu - n, \frac{x}{2K}, \frac{x}{2}) = \int_{\frac{x}{2K}}^{\frac{x}{2}} u^{-n}u^{\nu-1}e^{-u}du, \quad \text{(Generalized Incomplete Gamma Function)}
\]

\[
a_n (A, B) = (-1)^n C (B, n) A^n ,
\]

\[
C (B, n) = \prod_{k=1}^{n} \frac{(B - (k - 1))}{n!} 1_{\{n \geq 1\}} + 1_{\{n = 0\}} .
\]

If \( \nu - n \notin -N \) for each integer \( n \geq 0 \) (that is, for \( \rho \notin \{1/2, 3/4, 5/6, \ldots\} \)),
then

\[
E^Q \left[ \exp(-\lambda \tau_0) 1_{\{\tau_0 \leq T\}} | S \right] = \sum_{n=0}^{\infty} a_n (A, B) \left( \frac{x}{2} \right)^n \frac{\Gamma(\nu - n, \frac{x}{2K}, \frac{x}{2})}{\Gamma(\nu)} .
\]

For \( \lambda = 0 \), the well known \( Q \)-probability of a 100% drop within time \( T \) is


\[ E^Q \left[ 1_{\{\tau_0 \leq T\}} \mid S \right] = \frac{\Gamma(\nu, H)}{\Gamma(\nu)}, \]

where

\[ \Gamma(\nu, H) = \int_{H}^{+\infty} u^{\nu-1} e^{-u} du. \quad \text{(Incomplete Gamma Function)} \]

The Generalized Incomplete Gamma Function, the Incomplete Gamma Function, and the Gamma function are built-in routines in many computing software like MATLAB and Mathematica, which renders the above expressions fully viable.

Proposition 2 prices the Benchmark EDS fee quoted per annum as a fraction of the notional.

**Proposition 2** Under the CEV assumption and given \( k \) fee payments equally spaced within the year, the no-arbitrage fee of a Benchmark EDS with maturity \( T \ (T \in \mathbb{N} / 2, k \in \mathbb{N} / \{0\}) \) is

\[
f_{CEV} = \frac{E^Q \left[ \exp \left( -r \tau_0 \right) 1_{\{\tau_0 \leq T\}} \mid S \right] \cdot 50\%}{\sum_{j=1}^{kT} \frac{1}{k} \exp (-r T_j) \left( 1 - E^Q \left[ 1_{\{\tau_0 \leq T_j\}} \mid S \right] \right)}.
\]
where the \( T_j \)s are the \( \frac{1}{k} \)-spaced fee payment dates \((T_j \in \{1/k, 2/k, 3/k, \ldots, kT/k\})\) and the quantities \( E^Q \left[ \exp (-r \tau_0) 1_{\{\tau_0 \leq T\}} \mid S \right] \) and \( E^Q \left[ 1_{\{\tau_0 \leq T\}} \mid S \right] \) are calculated as in Proposition 1.

**Proof.** Under \( Q \), the transaction must result into a zero Net PV. The sum of the fee payment PVs (the accrual at \( \tau_0 \) of the last fee payment being disregarded) must equal the PV of the Benchmark EDS protection payment.

For a numerical inspection of the Benchmark EDS fee formula, consider semi-annual fee payments \((k = 2)\) and fix \( r = 5\% \) and \( q = 2\% \). The share price volatility parameter is

\[
\sigma = S^{1-\rho} \cdot 35\%,
\]

so that the reference entity’s share price has an initial volatility of 35\%. Setting \( \sigma \) in such a fashion also makes the \( f_{CEV} \) fee independent from the current share price. Table 1 exhibits the \( f_{CEV} \) fees\(^6\) (in basis points) across different maturities as well as across different intensities of the inverse relationship between volatility and share price. As the elasticity \( \rho - 1 \) becomes more negative, the CEV assumption is able to generate rich Benchmark EDS fees even for short maturities.

\(^6\)The first 20 series terms of the quantities in Proposition 1 are used.
Table 1: The Benchmark EDS fee. The parameter values are $k = 2$, $r = 5\%$, $q = 2\%$, and

$$\sigma = S^{1-\rho} \cdot 35\%.$$ 

<table>
<thead>
<tr>
<th>$\rho - 1$</th>
<th>$T = \frac{1}{2}$</th>
<th>$T = 1$</th>
<th>$T = 2.5$</th>
<th>$T = 5$</th>
<th>$T = 7.5$</th>
<th>$T = 10$</th>
</tr>
</thead>
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<tr>
<td>$-0.75$</td>
<td>000.0013</td>
<td>001.0819</td>
<td>042.6970</td>
<td>108.8289</td>
<td>132.1360</td>
<td>138.3188</td>
</tr>
<tr>
<td>$-1.00$</td>
<td>000.4694</td>
<td>018.6033</td>
<td>122.5820</td>
<td>179.8243</td>
<td>184.1596</td>
<td>177.7929</td>
</tr>
<tr>
<td>$-1.50$</td>
<td>033.4100</td>
<td>147.1834</td>
<td>269.7857</td>
<td>262.6213</td>
<td>234.7123</td>
<td>211.4019</td>
</tr>
<tr>
<td>$-2.00$</td>
<td>154.8685</td>
<td>313.5072</td>
<td>359.6726</td>
<td>296.7078</td>
<td>249.3659</td>
<td>217.1242</td>
</tr>
</tbody>
</table>

4 Conclusions

We employ a CEV equity market model to price in closed form the Benchmark EDS, a close equity-based counterpart of the CDS contract. This is done by deriving and applying a new result in the CEV asset pricing literature. The CEV assumption comes equipped with the ability of parsimoniously calibrating alternative credit-risk market information. Credit-related
analytic pricing under the CEV assumption offers a promising valuation outlook for hybrid corporate securities and for other hybrid financial products.
5 Appendix

The proof of Proposition 1 follows.

Proof. By Remark 2.1 and Corollary 3.1 in Delbaen and Shirakawa (2002), \( \tau_0 \) has the same \( Q \)-law as the random variable

\[
\frac{1}{2(r-q)(1-\rho)} \log \left( \frac{\sigma^2 (1-\rho)}{\sigma^2 (1-\rho) - 2(r-q) \tau_0} \right) \mathbb{1}_{\{\tau_0 < \frac{\sigma^2 (1-\rho)}{(r-q)}\}} + (\infty) \mathbb{1}_{\{\tau_0 \geq \frac{\sigma^2 (1-\rho)}{(r-q)}\}}
\]

\[
= \frac{B}{\lambda} \log \left( \frac{1}{1-A\tau_0} \right) \mathbb{1}_{\{\tau_0 < \frac{1}{A}\}} + (\infty) \mathbb{1}_{\{\tau_0 \geq \frac{1}{A}\}}
\]

where \( \tau_0 := \inf \left\{ s : X_s^{(2(1-\nu))} = 0 \right\} \) is the first hitting time at zero of the 2 \((1-\nu)\)-dimensional squared Bessel process, \( \{X_t^{(2(1-\nu))}\} \), starting at \( S^{2(1-\rho)} \).

Such a squared Bessel process has dynamics:

\[
dX^{(2(1-\nu))} = 2(1-\nu) dt + 2\left( |X^{(2(1-\nu))}| \right)^{\frac{1}{2}} dz.
\]

Since

\[
\frac{\sigma^2 (1-\rho)}{2(r-q)} > \frac{\sigma^2 (1-\rho)}{2(r-q)} \left( 1 - e^{-2(r-q)(1-\rho)T} \right),
\]

that is,

\[
\frac{1}{A} > K,
\]

the equivalence in law justifies the following statements:

\[
E^Q \left[ e^{-\lambda \tau_0} 1_{\{\tau_0 \leq T\}} \ | \ S \right] = E^Q \left[ e^{-B \log \left( \frac{1}{1-A\tau_0} \right) \mathbb{1}_{\{\tau_0 < \frac{1}{A}\}}} \mathbb{1}_{\{\tau_0 \leq K\}} \ | \ S \right] = E^Q \left[ (1-A\tau_0)^B \mathbb{1}_{\{\tau_0 < K\}} \ | \ S \right].
\]
Goeing-Jaesche and Yor (2003) - formula 28 - show that \( \hat{\tau}_0 \) has the following law:

\[
Q(\hat{\tau}_0 \in dt \mid S) = \frac{1}{t \Gamma(\nu)} \left( \frac{x}{2t} \right)^\nu e^{-\frac{x^2}{4t}} dt.
\]

We can write

\[
E_Q [e^{-\lambda \hat{\tau}_0 1_{\{\hat{\tau}_0 \leq T\}} \mid S] = \int_0^K (1 - At)^B \frac{1}{t \Gamma(\nu)} \left( \frac{x}{2t} \right)^\nu e^{-\frac{x^2}{4t}} dt.
\]

We perform the following power series expansion:

\[
(1 - At)^B = \sum_{n=0}^\infty (-1)^n C_n(B) A^n t^n,
\]

where \( C_n(B) \) is the \( n \)-th generalized binomial coefficient:

\[
C_n(B) = \frac{\Gamma(B) \Gamma(n+1)}{\Gamma(n+1-B)} 1_{\{n \geq 1\}} + 1_{\{n = 0\}}.
\]

We focus on \( t \) greater or equal of an arbitrarily small but strictly positive \( \epsilon \).

The series

\[
\frac{1}{\Gamma(\nu)} \left( \frac{x}{2} \right)^\nu t^{-1-\nu} \sum_{n=0}^\infty (-1)^n C_n(B) A^n t^n
\]

has a convergence radius of \( \frac{1}{A} \). Since

\[
\sum_{n=0}^\infty (-1)^n C_n(B) A^n t^n \frac{1}{t \Gamma(\nu)} \left( \frac{x}{2t} \right)^\nu e^{-\frac{x^2}{4t}} \leq \frac{1}{\Gamma(\nu)} \left( \frac{x}{2} \right)^\nu t^{-1-\nu} \sum_{n=0}^\infty (-1)^n C_n(B) A^n t^n,
\]

the left-hand-side series uniformly converges in \( t \in [\epsilon, K] \). Thus, we have

\[
E_Q [e^{-\lambda \hat{\tau}_0 1_{\{\hat{\tau}_0 \leq T\}} \mid S] = \lim_{\epsilon \downarrow 0} \int_\epsilon^K \sum_{n=0}^\infty (-1)^n C_n(B) A^n t^n \frac{1}{\Gamma(\nu)} \left( \frac{x}{2t} \right)^\nu e^{-\frac{x^2}{4t}} dt
\]

\[
= \lim_{\epsilon \downarrow 0} \sum_{n=0}^\infty (-1)^n C_n(B) A^n \left( \frac{x}{2} \right)^n \int_\epsilon^{\frac{x}{2}} \frac{1}{ \Gamma(\nu) u^{(\nu-n)} e^{-u} du
\]

\[
= \lim_{\epsilon \downarrow 0} \sum_{n=0}^\infty (-1)^n C_n(B) A^n \left( \frac{x}{2} \right)^n \frac{\Gamma(\nu-n, \frac{x}{2})}{\Gamma(\nu)}.
\]
If $\nu - n \notin -\mathbb{N}$ for each integer $n \geq 0$, the properties of the Incomplete Gamma Function imply that

$$E^Q [e^{-\lambda \tau_0} 1_{\{\tau_0 \leq T\}} \mid S] = \sum_{n=0}^{\infty} (-1)^n C_n (B) A^n \left( \frac{x}{2} \right)^n \frac{\Gamma (\nu - n, \frac{x}{2 \lambda})}{\Gamma (\nu)}.$$

The above condition on the parameter $\nu$ translates into $\rho \notin \{1/2, 3/4, 5/6, \ldots\}$.

If $\lambda = 0$, then $B = 0$ and we can easily recover the well known $Q$-probability of absorption at zero of the CEV model. Indeed, by setting $u = \frac{x}{2}$, we have

$$E^Q [1_{\{\tau_0 \leq T\}} \mid S] = \int_{H}^{+\infty} u^{\nu-1} e^{-u} \Gamma (\nu) - du,$$

where

$$H = \frac{(r - q) S^{2(1-\rho)}}{\sigma^2 (1 - \rho) [1 - e^{-2(r-q)(1-\rho)T}]}.$$
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