Investment under uncertainty with potential improvement of the operating cash flows

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Potential improvement of the operating cash flows from a project stimulates a firm to anticipate investment in the project. I characterize the optimal investment rule and derive the closed-form solution of the value of the option of waiting to invest. The major factors of investment forestalling are high project-value volatility, proximity to the project value that triggers cash flow improvement, and short *status-quo* operating cash flows.

JEL-Classification: G12, G13, G31.

Keywords: Investment under uncertainty, payout-rate switching.

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Abstract

Potential improvement of the operating cash flows from a project stimulates a firm to anticipate investment in the project. I characterize the optimal investment rule and derive the closed-form solution of the value of the option of waiting to invest. The major factors of investment forestalling are high project-value volatility, proximity to the project value that triggers cash flow improvement, and short *status-quo* operating cash flows.

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1 Introduction

How much should a firm anticipate investment in a project if, once the project has been entered[1], the firm will enjoy a form of cash flow insurance? The form of insurance I consider is an improvement of the operating cash flows from the project whenever business conditions deteriorate. Deteriorated business conditions are described by the project's value hitting a certain lower threshold. I carry out the analysis within the model of investment under uncertainty in McDonald and Siegel (1986), as reviewed in Dixit and Pindyck (1994) pp. 136-161. Such a model represents a classic bridge between real option theory and perpetual American option theory. I employ a double barrier contingent claims technique[2] to handle the free upper boundary for immediate investment as well as the lower threshold that triggers the improvement of the operating cash flows. My analysis shows that the major forces that drive anticipated investment are strong volatility in the project's value, project-value closeness to the lower threshold, and scarce *status-quo* operating cash flows.

The work is organized as follows. Section 2 introduces the investment problem with cash flow improvement. Section 3 looks at its dynamic programming formulation. Section 4 solves the problem by contingent claims analysis. Section 5 discusses the characteristics of the optimal investment rule. Section 6 concludes.

[1]For problems considering a firm that already enetered the project and faces a sequence of investment opportunities in project-technology innovations, see Grenadier and Weiss (1997).

[2]Sbuelz (2004) considers a different double barrier real option problem. He studies how much a firm will postpone investment if, under deteriorated business conditions, the sunk cost for accessing the project will decrease.

2 Investment under uncertainty with cash flow improvement

For a project with present value V, I consider the following problem: At what critical level $\overline{V^*}$ of the project's value is it optimal to pay the sunk cost I in return for the project's value itself, given that, as a fraction of the project's value, the operating cash flows over an infinitesimal time period dt will improve from $\delta \cdot dt$ to $\alpha \cdot \delta \cdot dt$, $\alpha \in [100\%, \infty)$, whenever V drops to a lower threshold L? The improvement of the project's operating cash flows is assumed to occur at a threshold that returns a negative net present value ($L < I$). The potential improvement of the project's operating cash flows is expressed by means of payout-rate switching. I assume that stochastic changes in V are spanned by existing traded assets of the economy. The spanning assumption will make the analysis akin to the treatment of perpetual American options. V evolves according to the following geometric Brownian motion (under the equivalent martingale measure):

$$dV = (r - \delta) V dt + \sigma V dz,$$

where r is the riskfree rate, δ is the *status-quo* payout rate, σ is the volatility rate, and dz is the increment of a Wiener process. The value of the option to invest is $F(V)$. It is the opportunity cost of investing now rather than waiting. $F_{\text{no-switch}}(V)$ is the value of the option to invest in the absence of payout-rate switching.

3 The dynamic programming problem

In the absence of any payout-rate switching, the firm is back to the McDonald and Siegel (1986) benchmark case. The benchmark critical value V^* is

$$\begin{aligned} V^* &= \frac{\beta_1}{\beta_1 - 1} I, \\ \beta_1 &= \frac{1}{2} - \frac{(r-\delta)}{\sigma^2} + \sqrt{\left[\frac{(r-\delta)}{\sigma^2} - \frac{1}{2}\right]^2 + \frac{2r}{\sigma^2}} > 1, \end{aligned}$$

and the firm's option to invest has the benchmark value

$$\begin{aligned} F_{\text{no-switch}}(V) &= \mathcal{E}\left[e^{-rT_{V^*}} \mid V\right] (V^* - I)\, 1_{\{V \le V^*\}} + (V - I)\, 1_{\{V^* < V\}}, \\ \mathcal{E}\left[e^{-rT_{V^*}} \mid V\right] &= \left(\frac{V}{V^*}\right)^{\beta_1}. \end{aligned}$$

$\mathcal{E}[\cdot \mid V]$ is the conditional expectation given the project's present value V. $\mathcal{E}\left[e^{-rT_{V^*}} \mid V\right]$ is the value of an American single barrier digital claim written on the project's value: One unit of numeraire is paid as soon as the project's value hits the level V^*. Since the project's value is spanned, the value of claims written on it is also spanned.

The Bellman equation for the payout-rate-switching problem is

$$F(V) = \max\left\{ \quad \max\{V - I, 0\} \quad , \quad \exp(-r\,dt)\,\mathcal{E}\left[F(V) + dF \mid V\right] \quad \right\},$$

with conditions,

$$
\begin{aligned}
F\left(\overline{V^*}\right) &= \overline{V^*} - I, && \text{(Value Matching Condition)} \\
\tfrac{d}{dV} F|_{V=\overline{V^*}} &= \tfrac{d}{dV}\left(V - I\right)|_{V=\overline{V^*}} = 1, && \text{(Smooth Pasting Condition)} \\
F\left(L\right) &= \mathcal{E}\left[e^{-rT_{V^{*,\alpha}}} \mid L\right]\left(V^{*,\alpha} - I\right), && \text{(Payout-Rate Switching Condition)}
\end{aligned}
$$

where

$$
\begin{aligned}
V^{*,\alpha} &= \frac{\beta_1\left(\alpha\right)}{\beta_1\left(\alpha\right) - 1} I, \\
\beta_1\left(\alpha\right) &= \frac{1}{2} - \frac{\left(r - \alpha\delta\right)}{\sigma^2} + \sqrt{\left[\frac{\left(r - \alpha\delta\right)}{\sigma^2} - \frac{1}{2}\right]^2 + \frac{2r}{\sigma^2}} > 1,
\end{aligned}
$$

In the region $\{V : V \in (0, L)\}$ the payout rate has already switched to the higher level $\alpha\delta$. There, the option value is

$$
F\left(V\right) = \mathcal{E}\left[e^{-rT_{V^{*,\alpha}}} \mid V\right]\left(V^{*,\alpha} - I\right).
$$

4 Solution by contingent claims analysis

In the region $\left\{V : V \in [L, \overline{V^*})\right\}$ there is no immediate investment. Thus, the Bellman equation boils down to the Black-Scholes Differential Equation (DE):

$$
\mathcal{E}\left[dF \mid V\right] = F\left(V\right) r dt.
$$

The value of any spanned contingent claim satisfies the Black-Scholes DE: In risk-adjusted expectation, the percentage increase in the spanned claim value must equal the claim's percentage cost of carry. Since the operator $\mathcal{E}\left[dF \mid V\right]$ is linear, any linear combination of spanned contingent claim values does satisfy the Black-Scholes DE. The boundary conditions introduce the free upper boundary $\overline{V^*}$ as well as the lower threshold L so that the firm is facing a double barrier problem. An educated guess suggests the use of a linear combination of American double barrier digital claims to build up the candidate solution. The values of the relevant American double barrier digitals are (see, for example, Geman and Yor (1996) and Lin (1998)):

$$\begin{aligned}
\mathcal{E}\left[e^{-rT_{\overline{V^*}}}1_{\{T_{\overline{V^*}}<T_L\}} \mid V\right] &= \frac{(V)^{\beta_1}(L)^{\beta_2}-(V)^{\beta_2}(L)^{\beta_1}}{\left(\overline{V^*}\right)^{\beta_1}(L)^{\beta_2}-\left(\overline{V^*}\right)^{\beta_2}(L)^{\beta_1}}, \\
\mathcal{E}\left[e^{-rT_L}1_{\{T_L<T_{\overline{V^*}}\}} \mid V\right] &= \frac{(V)^{\beta_2}\left(\overline{V^*}\right)^{\beta_1}-(V)^{\beta_1}\left(\overline{V^*}\right)^{\beta_2}}{\left(\overline{V^*}\right)^{\beta_1}(L)^{\beta_2}-\left(\overline{V^*}\right)^{\beta_2}(L)^{\beta_1}}, \\
\beta_2 &= \frac{1}{2}-\frac{(r-\delta)}{\sigma^2}-\sqrt{\left[\frac{(r-\delta)}{\sigma^2}-\frac{1}{2}\right]^2+\frac{2r}{\sigma^2}}<0.
\end{aligned}$$

The candidate solution $G(V)$,

$$\begin{aligned}
G(V) &= \mathcal{E}\left[e^{-rT_{\overline{V^*}}}1_{\{T_{\overline{V^*}}<T_L\}} \mid V\right]\left(\overline{V^*}-I\right) \\
&+\mathcal{E}\left[e^{-rT_L}1_{\{T_L<T_{\overline{V^*}}\}} \mid V\right]\mathcal{E}\left[e^{-rT_{V^{*,\alpha}}} \mid L\right]\left(V^{*,\alpha}-I\right),
\end{aligned}$$

satisfies by construction the Black-Scholes DE, the Value Matching Condition at the free boundary $\overline{V^*}$, and the Payout-Rate Switching Condition at the lower boundary L. The only condition left out is the Smooth Pasting Condition.

I force $G(V)$ to fulfill the Smooth Pasting Condition to generate a restriction that pins down the critical value [3] $\overline{V^*}$:

$$\frac{d}{dV}G|_{V=\overline{V^*}} = \frac{d}{dV}(V-I)|_{V=\overline{V^*}} = 1.$$

5 Characteristics of the optimal investment rule

I here examine the characteristics of the optimal investment rule and the value of the option to invest. In picking the parameter values, I rely on the classic calibration in Dixit and Pindyck (1994), p. 153. Unless otherwise noted, in what follows I set the cost of the investment, I, equal to 1, the riskfree rate $r = 4\%$, the *status-quo* payout rate $\delta = 4\%$, and the volatility rate $\sigma = 20\%$ (at annual rates). Given these parameter values, the benchmark critical value V^* is equal to $2I = 2$. If payout-rate switching is ruled out, V must be at least twice as large as the cost I before the firm should invest. Consider now a non-zero chance of a non-trivial payout-rate switch with parameters $L = 0.75$ and $\alpha = 500\%$. If the project's value drops down to 3/4 of the cost of undertaking it, the project's payout rate improves to go from its *status-quo* level to the new level of 20%. For these payout-rate-switching parameters, the critical value $\overline{V^*}$ is equal to $1.89 = 1.89I$. V must be less than twice as large as the *status-quo* cost I before the firm should invest. This anticipation of investment is driven by the fact that the chance of improving operating cash flows is greater if the firm enters the project at a critical value that is closer to L than the benchmark critical value V^* is. Figure 1 shows that L's proximity $(L \to I = 1)$

[3] Such a numerical computation of the critical value is easily performed in the *Mathematica* software because of $G(V)$'s analytic nature.

forces strong anticipation of investment.

As Figure 1 also shows, the sensitivities of the critical value to the project's value parameters have the same sign as the sensitivities of the benchmark critical value V^* (cf. Dixit and Pindyck (1994) pp. 152-161). The joint inspection of Figures 1, 2, 3, and 4 bears two considerations on the difference between the payout–rate-switching case with respect to the benchmark case. A high volatility rate σ does make a difference in the critical values. Such a significant anticipation of investment translates into an option value that is lower than the benchmark option value in a wide region of present values of the project. The result is driven by current-value propagation of a future riskier convex payoff and by the fact that, for any σ, the option value $F_{\text{no-switch}}(L)$ with $\delta = 4\%$ is greater than $F_{\text{no-switch}}(L)$ with $\alpha\delta = 20\%$. A high riskfree rate r does not make a strong difference in the critical values. The ensuing anticipation of investment translates into an option value that is lower than the benchmark option value in a confined region of present values of the project - the region between 0 and levels above but not far from L. The option value $F_{\text{no-switch}}(L)$ with $\delta = 4\%$ is greater than $F_{\text{no-switch}}(L)$ with $\alpha\delta = 20\%$ (and even more so as r increases) but, given $V > L$, the chance of payout-rate switching dims as the cost of carrying the project is pushed up by an increasing r. A high *status-quo* payout rate δ pulls down both critical values and stimulates early investment in any case, thus leaving no room for much difference. By contrast, a low *status-quo* payout rate δ does create a non-trivial difference in the critical values, as the prospect of a potential improvement of the scarse *status-quo* cash flows encourages anticipated investment. Figure 5 shows how a higher α magnifies the differences between the option values $F(V)$ and $F_{\text{no-switch}}(V)$. For example, a substantial improvement of the operating cash flows ($\alpha = 2000\%$) halves the value of the option of waiting to invest, on average, in the project-value region corresponding to $V \in (L = 0.75, I = 1)$ - this is achieved by pulling down the critical value $\overline{V^*}$ to the level $1.83 = 1.83I$.

6 Conclusions

How should a firm adjust the optimal timing of investment in a given project in the presence of potential project-technology switching? Sbuelz (2004) shows how investment-technology switching - the sunk cost for entering the project drops - induces postponement of investment. The present paper shows how the project-technolgy switching that takes the form of an improvement of the project's operating cash flows leads to anticipation of investment. The firm is willing to have early access to the project because the potential improvement in cash will yield benefits only once the firm has entered the project. Anticipation of investment is mainly driven by strong volatility in the project's value, by project-value closeness to the threshold that triggers project-technology switching, and by scarce *status-quo* operating cash flows. The limitations of the analysis are the assumption of an exogenous costless payout-rate improvement and the absence of industry equilibrium considerations. Addressing these two points makes an interesting agenda for future research.

References

Dixit, A. K., and R. S. Pindyck (1994), Investment under uncertainty, Princeton University Press, Princeton, New Jersey.

Geman, H., and M. Yor (1996), Pricing and hedging double-barrier options: a probabilistic approach, Mathematical Finance, 4, pp. 365-378.

Grenadier, A., and S. Weiss (1997), Investment in technological innovations: An option pricing approach, Journal of Financial Economics, 44, pp. 397-416.

Lin, X. (1998), Double barrier hitting time distributions with applications to exotic options, Insurance: Mathematics and Economics, 23, pp. 45-58.

McDonald R., and D. Siegel (1986), The value of waiting to invest, Quarterly Journal of Economics, pp. 707-727.

Sbuelz, A. (2004), The premium for switching investment technology when business conditions worsen, Working paper, Tilburg University.

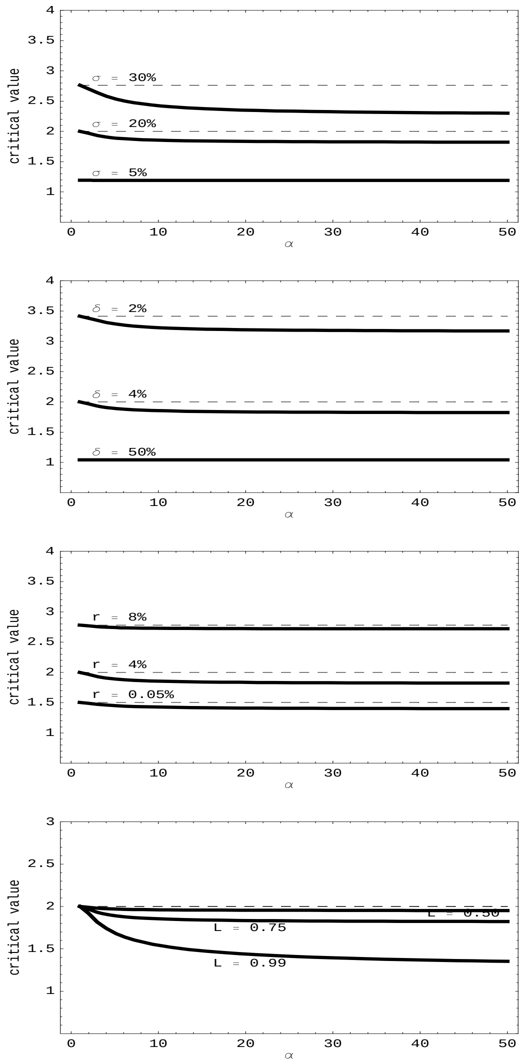

Figure 1: The critical value $\overline{V^*}$ versus the ratio α of new payout rate to *status-quo* payout rate

If not changed, these are the parameter values: The investment cost is $I = 1$; The riskfree rate is $r = 4\%$; The *status-quo* payout rate is $\delta = 4\%$; The project-value volatility is $\sigma = 20\%$; The lower threshold that triggers payout-rate switching is $L = 0.75$.

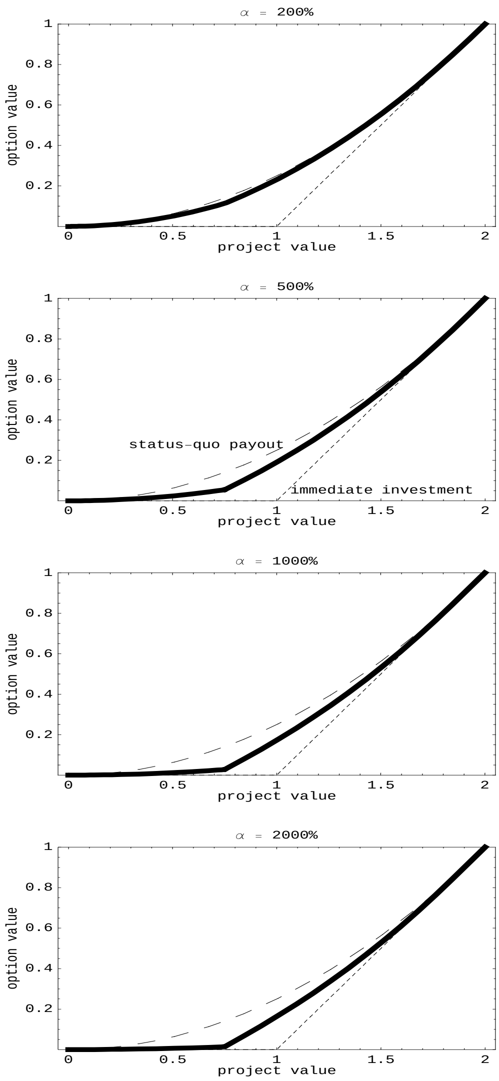

Figure 2: The investment-option value $F(V)$: The effect of the ratio α of new payout rate to *status-quo* payout rate

These are the parameter values: The investment cost is $I = 1$; The riskfree rate is $r = 4\%$; The *status-quo* payout rate is $\delta = 4\%$; The project-value volatility is $\sigma = 20\%$; The lower threshold that triggers payout-rate switching is $L = 0.75$.

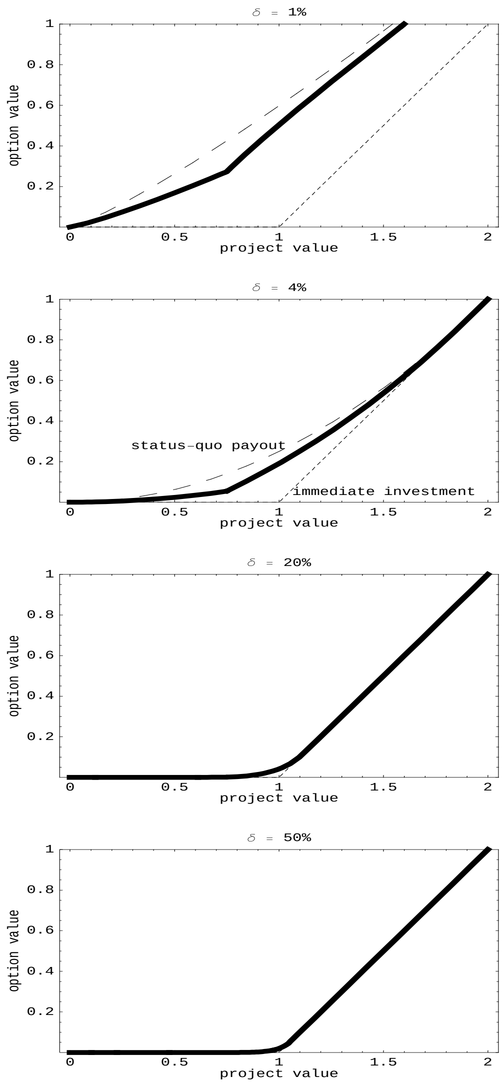

Figure 3: The investment-option value $F(V)$: The effect of the *status-quo* payout rate δ

These are the parameter values: The investment cost is $I = 1$; The riskfree rate is $r = 4\%$; The ratio of new payout rate to *status-quo* payout rate is $\alpha = 500\%$; The project-value volatility is $\sigma = 20\%$; The lower threshold that triggers payout-rate switching is $L = 0.75$.

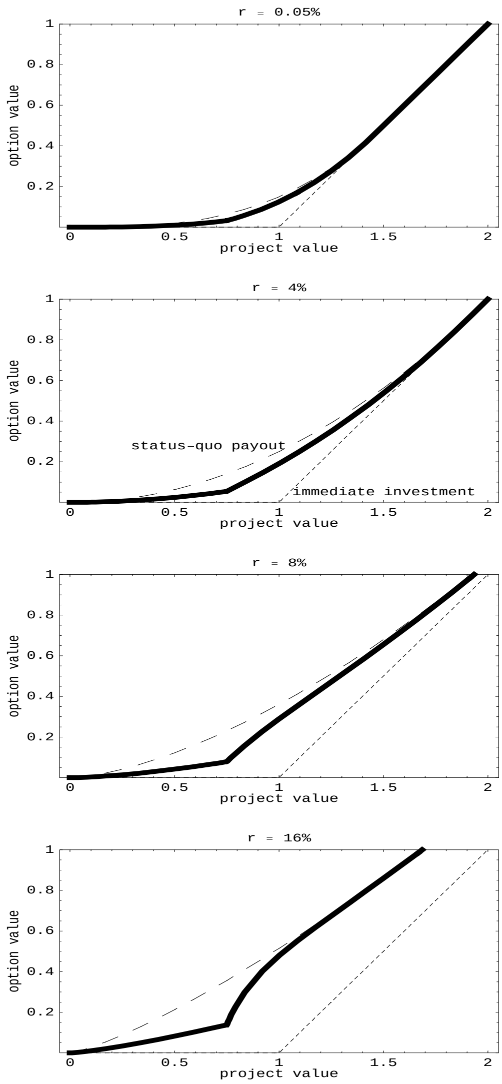

Figure 4: The investment-option value $F(V)$: The effect of the riskfree rate r

These are the parameter values: The investment cost is $I = 1$; The *status-quo* payout rate is $\delta = 4\%$; The ratio of new payout rate to *status-quo* payout rate is $\alpha = 500\%$; The project-value volatility is $\sigma = 20\%$; The lower threshold that triggers payout-rate switching is $L = 0.75$.

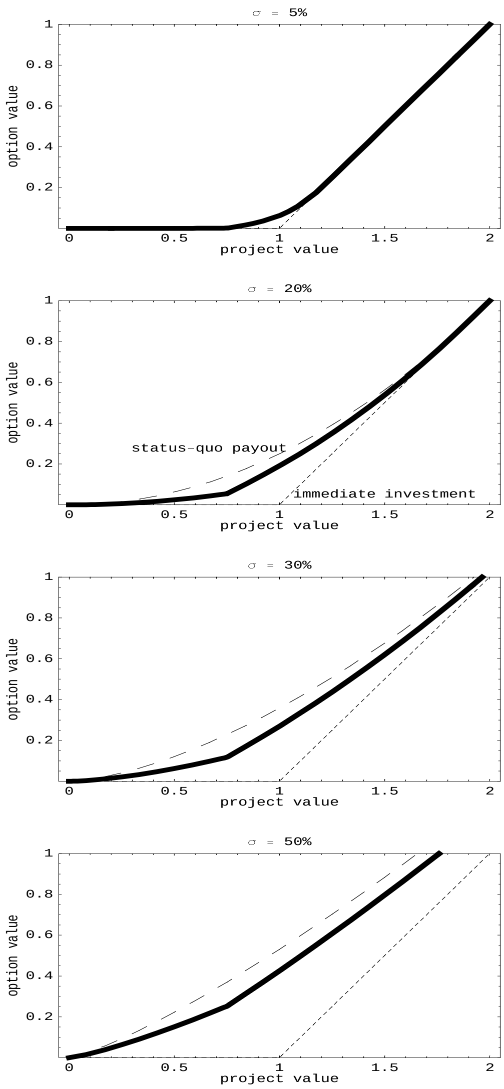

Figure 5: The investment-option value $F(V)$: The effect of the project-value volatility σ

These are the parameter values: The investment cost is $I = 1$; The riskfree rate is $r = 4\%$; The *status-quo* payout rate is $\delta = 4\%$; The ratio of new payout rate to *status-quo* payout rate is $\alpha = 500\%$; The lower threshold that triggers payout-rate switching is $L = 0.75$.